

Local tropicalization

Patrick Popescu-Pampu and Dmitry Stepanov

ABSTRACT. In this paper we propose a general functorial definition of the operation of *local tropicalization* in commutative algebra. Let R be a commutative ring, Γ a finitely generated subsemigroup of a lattice, $\gamma : \Gamma \rightarrow R/R^*$ a morphism of semigroups, and $\mathcal{V}(R)$ the topological space of valuations on R taking values in $\mathbb{R} \cup \infty$. Then we may *tropicalize* with respect to γ any subset \mathcal{W} of the space of valuations $\mathcal{V}(R)$. By definition, we get a subset of a rational polyhedral cone canonically associated to Γ , enriched with strata at infinity. In particular, when R is a local ring, γ is a *local* morphism of semigroups, and \mathcal{W} is the space of valuations which are either positive or non-negative on R , we call these processes *local tropicalizations*. They depend only on the ambient toroidal structure, which in turn allows to define tropicalizations of subvarieties of toroidal embeddings. We prove that with suitable hypothesis, these local tropicalizations are the supports of finite rational polyhedral fans enriched with strata at infinity and we compare the global and local tropicalizations of a subvariety of a toric variety.

CONTENTS

| | |
|---|----|
| 1. Introduction | 2 |
| 2. Geometry of semigroups | 5 |
| 3. Toric varieties | 10 |
| 4. Linear varieties associated to semigroups | 12 |
| 5. Valuation spaces | 18 |
| 6. An affine theory of tropicalization | 22 |
| 7. Extensions of valuations | 25 |
| 8. The formal toric rings $K[[\Gamma]]$ | 30 |
| 9. Standard bases | 34 |
| 10. Tropical bases | 40 |
| 11. The local finiteness theorem | 43 |
| 12. Comparison between local and global tropicalization | 51 |
| 13. Toroidal meaning of local tropicalization | 56 |

2010 *Mathematics Subject Classification.* Primary 14T05; Secondary 32S05, 14M25.

Key words and phrases. Bieri-Groves sets, semigroups, singularities, toric geometry, toroidal embeddings, tropicalization, valuation spaces.

The second author was supported by the Russian Grants for Leading Scientific Schools no. 5139.2012.1, and RFBR grant no. 11-01-00336-a.

| | |
|---|----|
| 14. An extension of the definition of tropicalization | 61 |
| 15. Comparison with the literature | 62 |
| References | 65 |

1. Introduction

Let V be a subvariety of a torus $(K^*)^n$ over an algebraically closed field K endowed with a non-trivial valuation $v : K \rightarrow \mathbb{R} \cup \{\infty\}$. Denote by I_V the ideal of V in the ring $K[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ of Laurent polynomials. Denote by x_1, \dots, x_n the images in the ring $K[V]$ of the canonical coordinate functions X_1, \dots, X_n on $(K^*)^n$.

If \mathbb{R}^n is the real vector space generated by the lattice \mathbb{Z}^n of 1-parameter subgroups of the torus $(K^*)^n$, we may also think about a vector $w \in \mathbb{R}^n$ as a weight of the variables X_1, \dots, X_n . Then, by definition, the w -initial ideal $\text{in}_w(I_V)$ is generated by all w -initial forms of elements of I_V (see also Section 8).

One associates canonically to $V \subset (K^*)^n$ a polyhedral set (that is, a set that may be represented as a finite union of convex polyhedra) in \mathbb{R}^n . This set is called *the tropicalization* of X . It can be defined in at least three different but equivalent ways, expressed as conditions (1), (2) and (3) in the following theorem (see [Sp], [SS], [EKL], [Pay07], [D]):

THEOREM 1.1. *The following subsets of \mathbb{R}^n coincide (the horizontal bar meaning the closure with respect to the usual topology of \mathbb{R}^n):*

- (1) $\overline{\{(v(s_1), \dots, v(s_n)) \mid (s_1, \dots, s_n) \in V\}}$.
- (2) $\overline{\{w \in \mathbb{R}^n \mid \text{the ideal } \text{in}_w(I_V) \text{ is monomial free}\}}$.
- (3) $\overline{\{(W(x_1), \dots, W(x_n)) \mid W \text{ is a valuation of } K[V] \text{ extending } v\}}$.

Our initial aim was to define *a local analog of tropicalization*, adapted to the study of singularities and of their deformations. More precisely, we wanted to tropicalize ideals of formal power series rings of the form $K[[X_1, \dots, X_n]]$, where K is any field, and to compare them with the previous (global) tropicalizations.

As subvarieties of tori are most naturally studied by taking their closures in associated toric varieties, we wanted to be able to define, more generally, tropicalizations of ideals in formal completions of the coordinate rings of affine toric varieties at their closed orbits, that is, in rings of the form $K[[\Gamma]]$, where Γ is a (not necessarily saturated) finitely generated subsemigroup of a lattice. In the sequel, following [CLS], we call such semigroups *affine*. In order to get more geometric flexibility (see Remark 7.7), we consider not necessarily normal toric varieties, that is, not necessarily saturated semigroups.

In order to compare local and global tropicalizations, we have to change the ring defining the object under study. That is why we need to develop a sufficiently general functorial framework for tropicalization. Among the characterizations 1–3 in the previous definition of (global) tropicalization, it is the third one which lends itself most easily to such a functorial treatment. This is not surprising since the set described by (3) is an image of Berkovich's

analytification of V , see [Berk]. Therefore, we propose the following general framework for tropicalization (both local and global):

- Start from a semigroup morphism $(\Gamma, +) \xrightarrow{\gamma} (R, \cdot)$ from an affine semigroup to the multiplicative semigroup of an arbitrary commutative ring.
- Consider the space $\mathcal{V}(R)$ of valuations of the ring R with values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$.
- Consider the tautological map:

$$\begin{array}{ccc} \mathcal{V}(R) & \xrightarrow{\gamma^*} & L(\Gamma) := \text{Hom}(\Gamma, \overline{\mathbb{R}}) \\ \nu & \longmapsto & \nu \circ \gamma. \end{array}$$

- If \mathcal{W} is any subset of $\mathcal{V}(R)$, its *tropicalization* is defined as the image $\gamma^*(\mathcal{W})$.

This construction is a functor from the category of pairs (γ, \mathcal{W}) and commutative diagrams of morphisms between such pairs to that of maps $\mathcal{W} \rightarrow L(\Gamma)$ and commutative diagrams between them.

We speak about *local tropicalization* when (R, \mathfrak{m}) is a local ring, γ is a local morphism (that is, $\gamma^{-1}(\mathfrak{m})$ is the set of non-invertible elements of Γ), and \mathcal{W} is a subset of the space of valuations centered on R (that is, nonnegative on R). There are two main instances of local tropicalization. The *positive tropicalization* of R with respect to a local morphism γ is the tropicalization of the space \mathcal{W} of valuations which are strictly positive on the maximal ideal \mathfrak{m} of R . The *nonnegative tropicalization* is defined similarly, with the only difference that we tropicalize all nonnegative valuations on \mathfrak{m} .

We consider the following particular instances of the previous definition:

- $\Gamma = \mathbb{Z}^n$, $R = K[V]$ where V is an algebraic subvariety of the torus $(K^*)^n$, γ is the natural morphism which sends each basis vector e_i of \mathbb{Z}^n to the image x_i in $K[V]$ of the corresponding variable X_i , and \mathcal{W} is the set of valuations extending the given one on K . Therefore, as a special case of our definition, we get the third version of the definition of the tropicalization of a subvariety of a torus as in Theorem 1.1.
- Γ is an arbitrary saturated affine semigroup, $R = K[V]$, V being an algebraic subvariety of the affine toric variety $\text{Spec } K[\Gamma]$ defined by Γ over K and \mathcal{W} is the whole space $\mathcal{V}(R)$. We get then the notion of tropicalization of a subvariety of a normal affine toric variety introduced by Payne [Pay08].

Our definition of local tropicalization can be applied in the following new setting:

- We let I be an ideal of a power-series ring $K[[\Gamma]]$, $R := K[[\Gamma]]/I$, γ be the natural semigroup morphism associating to each element of Γ the image in R of the corresponding monomial, and \mathcal{W} be the subspace of $\mathcal{V}(R)$ of valuations centered at R which extend the trivial valuation of K .

Our main structural results about tropicalization state the piecewise-linear structure of the local positive tropicalization (see Theorem 11.9 and Proposition 12.3 for the general statements). To give the reader an idea of

these results, we state here a particular case. Let us take $\Gamma = \mathbb{Z}_{\geq 0}^n$. Then $K[[\Gamma]]$ is isomorphic to the ring of formal power series in n variables.

THEOREM 1.2. *Let I be an ideal of the ring $K[[X_1, \dots, X_n]]$ of formal power series in n variables over an arbitrary field K endowed with the trivial valuation. Then:*

- (1) *The finite part of the local positive tropicalization $\text{Trop}_{>0}(I)$ of the ideal I (that is, of the natural morphism from Γ to the quotient local ring $K[[X_1, \dots, X_n]]/I$) is the support of a finite rational polyhedral fan in $(\mathbb{R}_+)^n$.*
- (2) *If I is prime and $K[[X_1, \dots, X_n]]/I$ has Krull dimension d , then $\text{Trop}_{>0}(I)$ has pure dimension d .*
- (3) *If I is the formal completion of the localization at 0 of an ideal J of the polynomial ring $K[X_1, \dots, X_n]$, then the local positive tropicalization $\text{Trop}_{>0}(I)$ coincides with the global tropicalization $\text{Trop}(J)$ of the subvariety of the torus defined by J inside the open cone $(\mathbb{R}_{>0})^n$.*

The last point of the theorem shows that it is possible to reconstruct the (global) tropicalization of a subvariety of a torus from local tropicalizations of its closure at the closed points of various toric varieties associated to that torus. In this sense, global tropicalization depends only on the boundary structure of the subvariety of the torus. In fact, the local tropicalization of an ideal I of $K[[\Gamma]]$ depends only on the toroidal structure of the ambient space $\text{Spec } K[[\Gamma]]$. In order to show this, we prove that, more generally, we can tropicalize semigroup morphisms of the form:

$$(\Gamma, +) \xrightarrow{\gamma} (R, \cdot)/(R^*, \cdot),$$

where (R^*, \cdot) denotes the subgroup of invertible elements of (R, \cdot) . This allows, e. g., to tropicalize objects which are not necessarily endowed with a toroidal structure:

- If $(X, 0)$ is a germ of normal (algebraic or analytic) variety and D is a reduced Weil divisor on it, consider a finitely generated semigroup Γ of effective Cartier divisors supported on D . Then, taking $R = \mathcal{O}_{X,0}$, we have a natural semigroup morphism $\Gamma \rightarrow R/R^*$, obtained by associating to each Cartier divisor a defining function in R , which is well-defined modulo units.
- We keep the same setting as in the previous example and let Γ be the full semigroup of effective Cartier divisors supported on D . Then we obtain a canonical tropicalization for each ideal I of $\mathcal{O}_{X,0}$ associated to the pair (X, D) , by taking $R := \mathcal{O}_{X,0}/I$ and the natural semigroup morphism $\Gamma \rightarrow R/R^*$ given by composing the map of the previous example with the map $\mathcal{O}_{X,0}/\mathcal{O}_{X,0}^* \rightarrow R/R^*$ induced by the quotient morphism $\mathcal{O}_{X,0} \rightarrow \mathcal{O}_{X,0}/I$.

These examples should be useful for the local study of Weil divisors on algebraic or analytic varieties, in such simple cases as those of germs of plane curves. In particular, they should allow to understand tropically a good amount of combinatorial invariants of singularities, for instance those extracted from weighted dual graphs of resolutions or embedded resolutions.

Each section in this paper begins with a brief description of its content. The comparison with the existing literature on the subject is concentrated in the last section, which also contains a brief description of possible interactions with developing fields of mathematics and two open problems.

Acknowledgments: The first author has benefited a lot from conversations with Angélica Cueto, Charles Favre and Anders Jensen and the second one from discussions with Mark Spivakovsky. We are grateful to Pedro D. González Pérez, Hussein Mourtada and Bernard Teissier for their remarks on previous versions of this paper. Last but not least, we thank heartedly the two anonymous referees for their very careful reading and for their many suggestions for clarification.

2. Geometry of semigroups

In this section we introduce the vocabulary and basic facts about semigroups that we shall use in this paper.

DEFINITION 2.1. A **semigroup** is a set Γ endowed with an associative binary operation $+: \Gamma \times \Gamma \rightarrow \Gamma$.

In the sequel, we shall consider only commutative semigroups. The simplest examples are abelian groups, but semigroups are interesting precisely because of the existence of elements which are not invertible.

If a semigroup has a neutral element 0, then we call it a semigroup *with origin*. If it has an ∞ element (also called *absorbing*), that is, an element which is unchanged by the addition of any other element, then it is called a semigroup *with infinity*. We see immediately that, if they exist, then the origin and the infinity are unique.

REMARK 2.2. If the semigroup law is thought multiplicatively then, by analogy with (\mathbb{R}, \cdot) , the origin is denoted 1 and the infinity is denoted 0 (see for example [How]), and we speak sometimes about semigroups *with identity* and *with zero*. Nevertheless, in the sequel we are consequent with the previous terminology and we say that, when $(R, +, \cdot)$ is a ring, then 0 is *the infinity* of the semigroup (R, \cdot) .

If a semigroup Γ has no origin, then we may canonically add such element to it, obtaining the semigroup with origin Γ_0 . If it has no infinity, we can analogously add to it a new element ∞ , getting $\bar{\Gamma}$.

DEFINITION 2.3. A semigroup is called **cancellative** if, whenever $a, b, c \in \Gamma$ satisfy $a + b = a + c$, we have $b = c$. It is called **of finite type** if it can be generated by a finite number of elements. It is called **torsion-free** if whenever $a, b \in \Gamma$ and $ma = mb$ for some $m \in \mathbb{N}^*$, we have $a = b$.

Note that a semigroup with infinity is not cancellative, excepted in the degenerate case when it has only one element, which is necessarily both the origin and the infinity. The following type of semigroups will play an essential role in our paper:

DEFINITION 2.4. A semigroup with origin $(\Gamma, +)$ is called **affine** if it is commutative, of finite type, cancellative and torsion-free.

The simplest affine semigroups are the various $(\mathbb{N}^n, +)$. The terminology “*affine*” is motivated by the fact that those are precisely the semigroups associated to *affine* toric varieties (see the next section and [CLS]).

Consider a semigroup Γ with origin. If $a \in \Gamma$, an *inverse* of a is an element $b \in \Gamma$ such that $a + b = 0$. If it exists, the inverse of a is unique and we denote it simply by $-a$. The set of invertible elements is a subgroup of Γ , which we denote Γ^* . We let Γ^+ be its complement in Γ . It is a prime ideal of Γ (see Lemma 2.7), in agreement with the next definition:

DEFINITION 2.5. If Γ is a semigroup, an **ideal** of Γ is a subset $I \subset \Gamma$ satisfying $I + \Gamma \subset I$. An ideal is called **proper** if $I \neq \Gamma$. The ideal I is **prime** if it is proper and, whenever $a, b \in \Gamma$ satisfy $a + b \in I$, then at least one of a, b is in I .

This vocabulary is motivated by the following fundamental example of semigroups:

EXAMPLE 2.6. Let $(R, +, \cdot)$ be a commutative ring. Forgetting the addition, (R, \cdot) is a semigroup with origin $1 \in R$. $(R \setminus \{0\}, \cdot)$ is cancellative if and only if R is a domain. Any ideal I of the ring $(R, +, \cdot)$ is an ideal of the semigroup (R, \cdot) . The converse is not true, as we do not ask for stability of the operation $+$ in the semigroup-theoretical definition of an ideal. For example, if $R = \mathbb{Z}$, the semigroup-ideal generated by 2 and 3 is the set of integers divisible either by 2 or by 3, which is not a ring-ideal.

LEMMA 2.7. *The subsemigroup Γ^+ of non-invertible elements of Γ is a prime ideal of Γ .*

PROOF. Let us first verify that Γ^+ is an ideal. Suppose that $a \in \Gamma^+$ and $b \in \Gamma$ satisfy $a + b \in \Gamma^*$. This means that there exists $c \in \Gamma$ such that $(a + b) + c = 0$. But this can be rewritten by associativity as $a + (b + c) = 0$, which shows that $a \in \Gamma^*$, a contradiction. Therefore, $\Gamma^+ + \Gamma \subset \Gamma^+$, which is the definition of the fact that Γ^+ is an ideal. The fact that this ideal is prime is immediate, consequence of the fact that Γ^* is stable under addition. \square

It is a formal exercise to see that the preimage of an ideal by a morphism of semigroups is again an ideal and that, moreover, in this way, prime ideals are transformed into prime ideals. Notice also that each semigroup Γ with origin is *local*, in the sense that it contains a unique maximal ideal Γ^+ .

In ring theory, ideals are precisely the kernels of the ring-morphisms. This is not true for semigroups. In order to speak about this phenomenon, we introduce basic notation about morphisms of semigroups. If Γ_1 and Γ_2 are semigroups, we denote by:

$$\text{Hom}_{Sg}(\Gamma_1, \Gamma_2)$$

the set of morphisms of semigroups from Γ_1 to Γ_2 . Analogously, if R_1 and R_2 are two rings, we denote by:

$$\text{Hom}_{Rg}(R_1, R_2)$$

the set of ring-morphisms.

If both semigroups Γ_1 and Γ_2 have origins, we assume that a morphism of semigroups sends one origin into the other. $\text{Hom}_{Sg}(\Gamma_1, \Gamma_2)$ has also naturally a structure of semigroup, by pointwise addition of the values.

Let $\phi : \Gamma_1 \rightarrow \Gamma_2$ be a morphism of semigroups with origins. Its set-theoretic image $\text{Im}(\phi)$ is a subsemigroup of Γ_2 and its kernel $\ker(\phi) := \phi^{-1}(0)$ is a subsemigroup of Γ_1 (in general it is not an ideal; that is, from this viewpoint, semigroups behave more like groups than like rings). Nevertheless, unlike for abelian groups, the knowledge of this kernel is not enough to determine the associated surjective map $\bar{\phi} : \Gamma_1 \rightarrow \text{Im}(\phi)$ up to isomorphism. Indeed, the fact that not all elements are invertible does not allow to conclude from $\phi(a) = \phi(b)$ that a is obtained from b by adding an element of the kernel. Briefly said, in general the kernel does not describe *all* the fibers of the map ϕ .

In order to be able to reconstruct the whole map $\bar{\phi}$, we have to encode the whole collection of its fibers. This may be done by looking at them as the equivalence classes of an equivalence relation \sim . This equivalence relation on Γ_1 is *compatible with the addition*, so it is a *congruence*:

DEFINITION 2.8. Let $(\Gamma, +)$ be a semigroup with origin. A **congruence** on Γ is an equivalence relation compatible with the addition.

If \sim is a congruence on Γ , we see immediately that the addition on Γ descends naturally to a semigroup law on the quotient Γ / \sim , the quotient map becoming a morphism of semigroups with origin.

For instance, the relation defined by

$$a \sim b \Leftrightarrow \exists c \in \Gamma^* \text{ such that } a = b + c$$

is a congruence. It allows to define the quotient semigroup Γ / Γ^* .

To any commutative semigroup Γ with origin, one functorially associates *the group* $M(\Gamma)$ generated formally by the differences of its elements:

$$a_1 - b_1 = a_2 - b_2 \Leftrightarrow \exists c \in \Gamma, a_1 + b_2 + c = a_2 + b_1 + c.$$

The canonical morphism of semigroups $\gamma : \Gamma \rightarrow M(\Gamma)$ is an embedding if and only if Γ is cancellative. Indeed, it is an embedding if and only if it is injective, which is equivalent to the fact that for any $a_1, a_2, c \in \Gamma$, the equality $a_1 + c = a_2 + c$ implies that $a_1 = a_2$. But this is precisely the condition of cancellation! For example, when $\Gamma = \mathbb{N}$ this gives the canonical inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$.

Assuming Γ to be cancellative, it is moreover torsion-free if and only if $M(\Gamma)$ is a torsion-free abelian group. Indeed, if there exists $n \in \mathbb{N}^*$ and $a \in \Gamma$ such that $n\gamma(a) = 0$, then there exists $c \in \Gamma$ such that $na + c = c$. As Γ is cancellative, we deduce that $na = 0$. As Γ is torsion-free, we conclude that $a = 0$.

On the other hand, it is not true that Γ is of finite type if and only if $M(\Gamma)$ is of finite type. For instance, $M((\mathbb{N}^*)^2) = \mathbb{Z}^2$ is of finite type but $(\mathbb{N}^*)^2$ is not of finite type. Only the following implication holds: if Γ is of finite type, then so is $M(\Gamma)$.

We define a *lattice* as an abelian torsion-free group of finite type. The previous explanations have as a direct consequence the following characterization of affine semigroups:

LEMMA 2.9. *A semigroup is affine if and only if it is a finitely generated subsemigroup of a lattice and it has an origin.*

Let Γ be an affine semigroup and $M(\Gamma)$ be its associated lattice. We denote by $N(\Gamma) := \text{Hom}_{\mathbb{C}p}(M(\Gamma), \mathbb{Z})$ the dual lattice.

DEFINITION 2.10. The **saturation** $\text{Sat}(\Gamma) \hookrightarrow M(\Gamma)$ of Γ (inside $M(\Gamma)$) is the subset of $M(\Gamma)$ formed by the elements v satisfying $nv \in \Gamma$ for some $n \in \mathbb{N}^*$. A semigroup is called **saturated** if it is equal to its saturation.

EXAMPLE 2.11. Let us consider the affine subsemigroup Γ of $\mathbb{N} \times \mathbb{Z}$ generated by $v_1 = (2, 1), v_2 = (5, 2), v_3 = (0, 3), v_4 = (0, -3)$ (see Figure 1). The associated lattice $M(\Gamma)$ is equal to \mathbb{Z}^2 , and $\text{Sat}(\Gamma)$ is $\mathbb{N} \times \mathbb{Z}$. As is visible in the drawing, Γ^* is the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by v_3 . In the drawing is also represented the quotient map $p : \Gamma \rightarrow \Gamma/\Gamma^*$. This last semigroup Γ/Γ^* is isomorphic to the image of Γ by the canonical projection of $\mathbb{N} \times \mathbb{Z}$ to the first factor \mathbb{N} . Therefore it is affine.

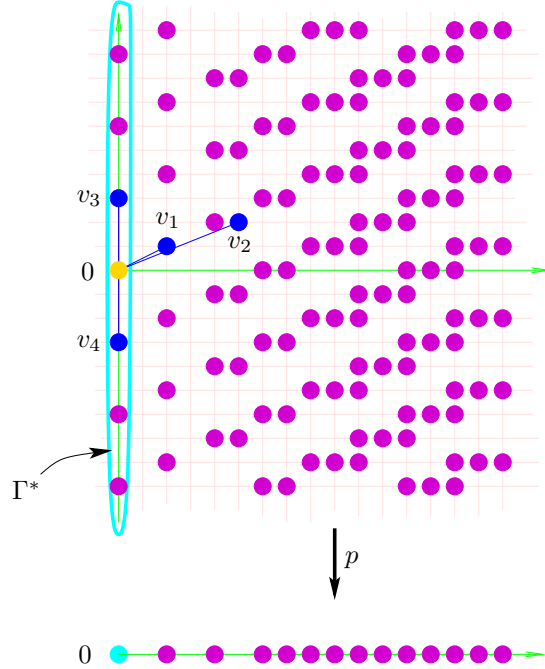


FIGURE 1. An affine semigroup and its quotient by its subgroup of invertible elements

EXAMPLE 2.12. In the previous example, the quotient Γ/Γ^* was again affine. This is not true for all affine semigroups Γ . Consider for instance the affine subsemigroup of \mathbb{Z}^2 generated by $v_1 = (1, 0), v_2 = (1, 1), v_3 = (0, 2), v_4 = (0, -2)$ (see Figure 2). Then Γ^* is the lattice of rank one generated by v_3 . The quotient Γ/Γ^* has torsion, as the images $\gamma(v_1)$ and $\gamma(v_2)$ are different (there does not exist any $c \in \Gamma^*$ such that $v_2 = v_1 + c$) but their doubles are equal (as $2v_2 = 2v_1 + v_3$). In fact, the restriction to Γ of the second projection $\mathbb{Z}^2 \rightarrow \mathbb{Z}$ factors through the quotient map p , inducing a map $\bar{p} : \Gamma/\Gamma^* \rightarrow \mathbb{N}$. The fibers $\bar{p}^{-1}(n)$ of this map have two points for $n > 0$, only the origin being covered by one point. That is why we represented the Γ/Γ^* as the set \mathbb{N} in which every positive number is split into two points.

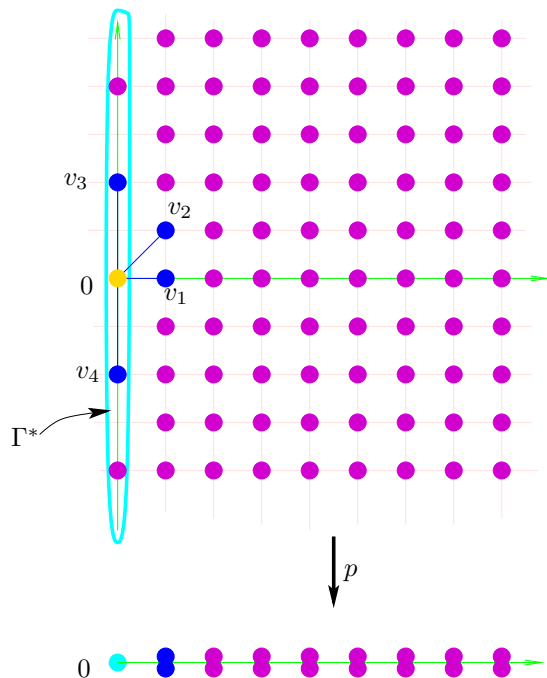


FIGURE 2. A quotient with torsion of an affine semigroup by its subgroup of invertible elements

PROPOSITION 2.13. *If the affine semigroup Γ is saturated, then Γ/Γ^* is also a saturated affine semigroup.*

Let us show that Γ' is *cancellative*. Suppose that $a, b, c \in \Gamma$ satisfy the equality $\overline{a} + \overline{b} = \overline{a} + \overline{c}$. This implies that there exists $d \in \Gamma^*$ such that $a + b = (a + c) + d = a + (c + d)$. As Γ is assumed cancellative, we deduce that $b = c + d$, which implies that $\overline{b} = \overline{c}$. That is, Γ' is also cancellative.

We show now that Γ' is *torsion-free*. Assume that $a, b \in \Gamma$ satisfy an equality of the type $n\bar{a} = n\bar{b}$, with $n \in \mathbb{N}^*$. Therefore, there exists $c \in \Gamma'$ such that $na = nb + c$. Inside the lattice $M(\Gamma)$ (into which Γ embeds canonically), we may write the previous equality as $n(a - b) = c$. Our hypothesis that Γ is saturated implies that there exists $d \in \Gamma$ such that $a - b = d$. The same argument repeated with the equality $n(b - a) = -c$ would give us a $d' \in \Gamma$ with $b - a = d'$. Then $d + d' = 0$, which shows that $d \in \Gamma^*$. We conclude that $\bar{a} = \bar{b}$. That is, Γ' is torsion-free.

Finally, let us show that Γ' is also *saturated*. The previous argument shows that Γ^* is a primitive sublattice of $M(\Gamma)$, that is, that the quotient $M(\Gamma)/\Gamma^*$ is torsion-free. As the images of the generators of Γ also generate this quotient, we deduce that $M(\Gamma)/\Gamma^*$ is canonically isomorphic to $M(\Gamma')$. We will work with this representative of the associated lattice. Consider

therefore $v \in M(\Gamma)$ such that there exists $n \in \mathbb{N}$ and $a \in \Gamma$ with $n\bar{v} = \bar{a}$ in $M(\Gamma)/\Gamma^*$. Therefore, there exists $c \in \Gamma^*$ such that $nv = a + c$. But $a + c \in \Gamma$ and Γ is saturated, which implies that $v \in \Gamma$. Therefore $\bar{v} \in \Gamma'$, which shows that Γ' is saturated. \square

Until now we have only discussed algebraic aspects of semigroups. We now describe their topology. Suppose that the semigroup Γ *has no infinity* and moreover *is totally ordered*, that is, it is endowed with a total order compatible with the addition. Then, we can equip Γ with a natural topology generated by the “open” intervals. We extend this topology to $\bar{\Gamma}$ by taking as basis of neighborhoods of ∞ the subsets of the form:

$$(a, \infty] := \{x \in \Gamma \mid x > a\} \cup \{\infty\}.$$

Note that with this convention, $\infty = +\infty$, not $-\infty$. We will mainly use the previous construction of topology when Γ is \mathbb{R} , $[0, \infty)$, $(0, \infty)$, \mathbb{Z} , \mathbb{N} , the semigroup operation being addition.

3. Toric varieties

This section is intended only to set the notations we use for toric geometry. For details on *normal* toric varieties, we refer to Fulton’s book [Ful]. For not necessarily normal toric varieties, the reader can consult the recent monograph [CLS] of Cox, Little, and Schenck, or Gonzalez Pérez and Teissier’s paper [GT].

In the sequel, if G is an abelian group and K is a field, we will denote by G_K the K -vector space $G \otimes_{\mathbb{Z}} K$.

Let Γ be an affine smigroup. We denote by:

$$\check{\sigma}(\Gamma) \subset M(\Gamma)_{\mathbb{R}}$$

the finite rational polyhedral cone generated by Γ . By definition, it consists of all the combinations with nonnegative real coefficients of elements of Γ . It is a sub-cone of the real vector space $M(\Gamma)_{\mathbb{R}}$ with non-empty interior. The saturation of Γ may be described geometrically using the cone $\check{\sigma}(\Gamma)$:

$$(3.1) \quad \text{Sat}(\Gamma) = \check{\sigma}(\Gamma) \cap M(\Gamma).$$

We denote by $\sigma(\Gamma) \subset N(\Gamma)_{\mathbb{R}}$ the dual cone, defined by:

$$\sigma(\Gamma) := \{w \in N(\Gamma)_{\mathbb{R}} \mid w(\check{\sigma}(\Gamma)) \subset \mathbb{R}_{\geq 0}\}.$$

More generally, if σ is a polyhedral cone in a finite dimensional real vector space V and $\check{\sigma}$ is its dual cone in the dual space V^* , then $\check{\check{\sigma}} = \sigma$ and the map:

$$\tau \longrightarrow \check{\sigma} \cap \tau^{\perp}$$

establishes an inclusion-reversing bijection between the closed faces of σ and those of $\check{\sigma}$.

The cone $\check{\sigma}$ has non-empty interior if and only if σ is *strictly convex*, that is, if it does not contain any vector subspace of positive dimension. We will say also in this case that σ is a *pointed cone*. If $\Gamma^* = 0$, we say that Γ is a *pointed semigroup*. It is immediate to check that the affine semigroup Γ is pointed if and only if the cone $\check{\sigma}(\Gamma)$ is pointed.

The vocabulary introduced in the following definition is taken from [GT]:

DEFINITION 3.1. If Γ is an affine semigroup, then a **face** of Γ is a sub-semigroup $\Lambda \subset \Gamma$ such that whenever $x, y \in \Gamma$ satisfy $x + y \in \Lambda$, then both x and y are in Λ .

The following proposition characterizes the faces of Γ :

PROPOSITION 3.2. *The faces of Γ are precisely the complements of the prime ideals of Γ . The map $\tau \longrightarrow \Gamma \cap \tau^\perp$ establishes an inclusion-reversing bijection between the faces of $\sigma(\Gamma)$ and those of Γ .*

In the sequel, if Γ is affine and τ is a face of $\sigma(\Gamma)$, we denote:

$$\Gamma_\tau := \Gamma \cap \tau^\perp, \quad M(\tau, \Gamma) := M(\Gamma_\tau), \quad M(\tau) := M \cap \tau^\perp.$$

If N_τ denotes the sublattice of N spanned by $\tau \cap N$, the quotient N/N_τ is canonically dual to $M(\tau)$, i.e., $N/N_\tau \simeq \text{Hom}(M(\tau), \mathbb{Z})$.

The subgroup Γ^* of invertible elements is the *minimal* face, in the sense that it is contained in all the other ones. By the previous bijection, it corresponds to $\tau = \sigma$.

Let Γ be an affine semigroup and K be a field. We denote by:

$$\mathcal{Z}_K(\Gamma) := \text{Spec } K[\Gamma]$$

the associated *toric variety* defined over K . Its *K -valued points* are naturally identified with the semigroup:

$$(3.2) \quad \text{Hom}_{Rg}(K[\Gamma], K) \simeq \text{Hom}_{Sg}((\Gamma, +), (K, \cdot)).$$

Notice that the multiplicative semigroup (K, \cdot) has 0 as infinity.

When Γ is of the form $\check{\sigma} \cap M$, and σ is a strictly convex rational polyhedral cone in the real vector space $N_{\mathbb{R}}$, we define:

$$\mathcal{Z}_K(\sigma, N) := \mathcal{Z}_K(\check{\sigma} \cap M).$$

These are precisely the *normal* affine toric varieties.

In the same way as abstract varieties over a field are obtained by gluing affine cones, we can glue affine toric varieties by respecting the ambient structure, that is, the action of the torus $T_K(N) := \text{Spec } K[M]$. This is easiest to describe in the case of normal toric varieties: the combinatorial object encoding the gluing is a *fan*.

DEFINITION 3.3. A **fan** in $N_{\mathbb{R}}$ is a finite set Δ of convex polyhedral cones inside $N_{\mathbb{R}}$, such that:

- a) for each cone σ in Δ , all its faces are in Δ ;
- b) if σ_1 and σ_2 are cones of Δ , then $\sigma_1 \cap \sigma_2$ is a common face.

If all the cones are rational, that is, they are defined as the intersections of halfspaces $\{v \in N_{\mathbb{R}} \mid \langle v, m \rangle \geq 0\}$, where $m \in M$ and $\langle \cdot, \cdot \rangle$ is the pairing between N and M , then the fan is called **rational**.

These conditions imply that all the cones in Δ have a maximal common linear subspace. We say that Δ is a *pointed fan*, if this linear subspace is the origin, that is, if all the cones of Δ are *strictly convex*.

If Δ is a pointed finite rational polyhedral fan inside $N_{\mathbb{R}}$, we denote by $\mathcal{Z}_K(\Delta, N)$ the normal toric variety over the field K associated to the lattice

N and the fan Δ . It is obtained by the usual gluing of affine toric varieties: $\mathcal{Z}_K(\sigma_1, N)$ and $\mathcal{Z}_K(\sigma_2, N)$ are glued along $\mathcal{Z}_K(\sigma_1 \cap \sigma_2, N)$, which is an open affine toric subvariety of both of them (see [Ful] or [CLS]).

There is an incidence-reversing bijection between the cones of Δ and the orbits of the torus action on $\mathcal{Z}_K(\Delta, N)$. We denote by O_σ the orbit associated to the cone $\sigma \in \Delta$. It is canonically identified with the torus $T_{N/N_\sigma, K}$.

We may also encode combinatorially the gluing of not necessarily normal affine toric varieties, as explained by González Pérez and Teissier in [GT]. For this, we propose the following new notion:

DEFINITION 3.4. A **fan of semigroups** \mathcal{S} in N is a rational fan Δ in $N_{\mathbb{R}}$, enriched with an affine subsemigroup Γ_α of Γ for each cone $\alpha \in \Delta$, with the property that:

- i) for each cone α in Δ , $\sigma(\Gamma_\alpha) = \alpha$ and $M(\Gamma_\alpha) = M$;
- ii) if β is a face of $\alpha \in \Delta$, then $\Gamma_\beta = \Gamma_\alpha + M(\beta, \Gamma_\alpha)$.

Note that condition i) implies that the fan Δ is pointed. These conditions allow to glue the affine toric varieties $\mathcal{Z}_K(\Gamma)$ corresponding to the various semigroups of a given fan of semigroups \mathcal{S} . We denote by $\mathcal{Z}_K(\mathcal{S})$ the associated toric variety.

4. Linear varieties associated to semigroups

In this section we develop a theory of embeddings of topological semigroups into bigger stratified topological spaces endowed with an action of the initial semigroup. This generalizes a construction introduced by Ash, Mumford, Rapoport, Tai [AMRT, I.1] and developed recently by Payne [Pay08] and Kajiwara [Kaj]. Their setting corresponds to the case when the semigroup is a strictly convex cone in a finite dimensional real vector space.

Recall that we assume all the semigroups to be commutative and with origin. Let G and H be semigroups, and denote by:

$$\mathcal{Z}_H(G) := \text{Hom}_{Sg}(G, H)$$

the semigroup of semigroup morphisms from G to H . We think about it as the set of H -valued points of the semigroup G . Moreover, when H is a topological semigroup, we endow $\mathcal{Z}_H(G)$ with the *topology of pointwise convergence*, that is, the induced topology coming from the natural embedding $\mathcal{Z}_H(G) \hookrightarrow H^G$, the target space being endowed with the product topology.

EXAMPLE 4.1. When G is an affine semigroup and H is the multiplicative group (K^*, \cdot) of a field K , then $\mathcal{Z}_H(G)$ equals the torus $T_K(N(G))$, whose lattice of characters is the lattice $M(G)$ associated to G . The torus $T_K(N(G))$ is naturally an algebraic variety and bears the Zariski topology. If there is some natural topology on K , the topology of pointwise convergence on $T_K(N(G))$ is different from the Zariski topology.

EXAMPLE 4.2. When G is either an affine semigroup or a polyhedral cone and H is the additive group $(\mathbb{R}, +)$, then $\mathcal{Z}_H(G)$ is the real vector space $N(G)_{\mathbb{R}}$. This notation was explained before in the case of affine semigroups.

When G is a cone $\check{\sigma}$, $N(G)_{\mathbb{R}}$ denotes the dual space to the vector space $M(\check{\sigma})$ generated by $\check{\sigma}$.

Notice that the functor $(G, H) \rightarrow \mathcal{Z}_H(G)$ is contravariant in the variable G and covariant in the variable H (this is, of course, valid in any category). When H has no infinity, we get in particular a natural embedding of semigroups:

$$\mathcal{Z}_H(G) \hookrightarrow \mathcal{Z}_{\overline{H}}(G).$$

DEFINITION 4.3. If G is a semigroup and H a semigroup without infinity, we say that $\mathcal{Z}_{\overline{H}}(G)$ is the **H -valued (affine) linear variety of G** .

REMARK 4.4. We chose this name in analogy with that of *toric varieties*. Indeed, when G is an affine semigroup and $H = (K^*, \cdot)$, as in Example 4.1, $\mathcal{Z}_{\overline{K^*}}(G) = \mathcal{Z}_{(K, \cdot)}(G) = \text{Hom}_{S_g}(G, (K, \cdot))$, is the set of K -valued points of the affine toric variety $\mathcal{Z}_K(G)$ (see formula (3.2)). The attribute “*toric*” makes reference to a natural action of an algebraic (split) torus, whose law is thought of multiplicatively. In our context, the analog of the torus is the semigroup $\mathcal{Z}_H(G)$, thought of additively. It acts naturally on the linear variety $\mathcal{Z}_{\overline{H}}(G)$. The most important case for us is $H = \mathbb{R}$, when $\mathcal{Z}_H(G)$ is a vector space. This explains the attribute “*linear*” in our terminology. In what concerns the attribute “*affine*”, it makes reference to the fact that we define an analog of the notion of *affine* toric variety.

Assume now that H is a group. In the same way as toric varieties are canonically stratified into the orbits of the associated torus, the linear variety $\mathcal{Z}_{\overline{H}}(G)$ is stratified into the orbits of the natural action of $\mathcal{Z}_H(G)$ on $\mathcal{Z}_{\overline{H}}(G)$ induced by the addition $H \times \overline{H} \rightarrow \overline{H}$ on the values. For an affine G and divisible H , these orbits may be described in a different way, using the notion of prime ideal of a semigroup (see Definition 2.5):

PROPOSITION 4.5. *Let H be a divisible group and G an affine semigroup. The orbits of the natural action of $\mathcal{Z}_H(G)$ on $\mathcal{Z}_{\overline{H}}(G)$ are in a bijection with the prime ideals of G . The bijection is given by:*

$$\text{the orbit of } \gamma \in \mathcal{Z}_{\overline{H}}(G) \quad \leftrightarrow \quad \text{the prime ideal } \gamma^{-1}(\infty \in \overline{H}).$$

Therefore, those orbits are in natural bijection with the faces of G (see Proposition 3.2). If \mathcal{S} is a fan of affine semigroups and H is a divisible group, then we have the canonical identification $\mathcal{Z}_H(\Gamma) = \mathcal{Z}_H(M(\Gamma))$, where $\Gamma \in \mathcal{S}$ and $M(\Gamma)$ is the same lattice for all the semigroups $\Gamma \in \mathcal{S}$, by condition i) of Definition 3.4.

REMARK 4.6. The construction of [AMRT, I.1] alluded to in the introductory paragraph of this section, and developed further by Payne [Pay08] and Kajiwara [Kaj], corresponds to the case when G is a saturated affine semigroup $\check{\sigma} \cap M$ and $H = \mathbb{R}$ (see again Example 4.1). We chose to develop this more general categorical viewpoint for the following reasons:

- (1) To get extra structures on G_H from the functorial properties of our construction. For instance, when G is affine, then, the integral points of $\mathcal{Z}_{\overline{\mathbb{R}}}(G)$ are the points in $\mathcal{Z}_{\overline{\mathbb{Z}}}(G)$.

- (2) To study also valuations taking values in totally ordered groups which do not embed into \mathbb{R} , that is, which have rank at least 2. For instance, this could be useful when developing the theory initiated by F. Aroca in [Ar].

We focus now on the topological aspects of the constructions. Let G be a saturated affine semigroup $\check{\sigma} \cap M$, $H = \mathbb{R}$, $\sigma \subset N_{\mathbb{R}}$ is a strictly convex rational polyhedral cone. Fix:

$$(4.1) \quad L(\sigma, N) := \text{Hom}_{Sg}(\check{\sigma} \cap M, \overline{\mathbb{R}}) = \mathcal{Z}_{\overline{\mathbb{R}}}(\check{\sigma} \cap M),$$

$$(4.2) \quad \overline{\sigma} = \overline{(\sigma, N)} := \text{Hom}_{Sg}(\check{\sigma} \cap M, \overline{\mathbb{R}}_{\geq 0}) = \mathcal{Z}_{\overline{\mathbb{R}}_{\geq 0}}(\check{\sigma} \cap M).$$

Whenever N is clear from the context, we omit it and denote $\overline{(\sigma, N)}$ simply by $\overline{\sigma}$. We denote by $\overline{\sigma}^{\circ}$ the subspace of $\overline{\sigma}$ consisting of those semigroup morphisms $\check{\sigma} \cap M \rightarrow \overline{\mathbb{R}}_{\geq 0}$ which take only positive values (possibly $+\infty$) on the maximal ideal of the semigroup $\check{\sigma} \cap M$. We say that $\overline{\sigma}^{\circ}$ is the *interior* of $\overline{\sigma}$.

We view $L(\sigma, N)$ as a space of functions from the set $\check{\sigma} \cap M$ to the topological space $\overline{\mathbb{R}}$, and endow it with the topology of pointwise convergence. This is the weakest topology for which all the sets $\{\gamma \in L(\sigma, N) \mid \gamma(m) \in U\}$ are open, where $m \in \check{\sigma} \cap M$ and U is an open subset of $\overline{\mathbb{R}}$. Since the topological space $\overline{\mathbb{R}}$ is separated, this topology is also separated. Since \mathbb{R} is a dense open subspace of $\overline{\mathbb{R}}$, we see that $N_{\mathbb{R}}$ is also a dense open subspace of $L(\sigma, N)$.

Respecting the conventions of Definition 4.3, we introduce the following terminology:

DEFINITION 4.7. The topological space $L(\sigma, N)$, endowed with the natural continuous action $N_{\mathbb{R}} \times L(\sigma, N) \rightarrow L(\sigma, N)$ extending the action of $N_{\mathbb{R}}$ on itself by translations is called **the affine linear variety** associated to the pair (N, σ) . We say that the closure of the cone σ in $L(\sigma, N)$ is the **extended cone** $\overline{\sigma}$.

The affine linear variety $L(\sigma, N)$ is obtained by adding to the vector space $N_{\mathbb{R}}$ some strata *at infinity*, each stratum being by definition an orbit of the previous action. These strata have a canonical structure of vector spaces, in the same way as the orbits of the canonical action of a torus on an associated affine toric variety are canonically lower-dimensional tori. More precisely, they are canonically identified with the vector spaces $(N/N_{\tau})_{\mathbb{R}}$, where τ is a face of the cone σ (including 0 and σ itself). Here, N_{τ} denotes the intersection of the vector space spanned by τ with the lattice N .

We now introduce a topology on the disjoint union $\bigsqcup_{\tau} (N/N_{\tau})_{\mathbb{R}}$. If U is an open subset of $N_{\mathbb{R}}$ and δ is a face of σ , we consider the set:

$$(4.3) \quad \overline{U}_{\delta} := \bigsqcup_{\tau \leq \delta} \pi_{\tau}(U + \delta) \subseteq \bigsqcup_{\tau \leq \sigma} (N/N_{\tau})_{\mathbb{R}}$$

where $N_{\mathbb{R}} \xrightarrow{\pi_{\tau}} (N/N_{\tau})_{\mathbb{R}}$ is the canonical projection, the first union is taken over all faces τ of the cone δ , and the second over those of σ .

The disjoint union $\bigsqcup_{\tau} (N/N_{\tau})_{\mathbb{R}}$ endowed with the previous topology is a partial compactification of $N_{\mathbb{R}}$. Intuitively, this topology may be explained as follows: *the sequence $(v_n)_{n \in \mathbb{N}} \subset N_{\mathbb{R}}$ tends to $v^{\tau} \in (N/N_{\tau})_{\mathbb{R}}$ if and only if $(v_n)_{n \in \mathbb{N}}$ tends to infinity in the direction of the cone τ and the sequence of projections $(p_{\tau}(v_n))_{n \in \mathbb{N}}$ converges to $v^{\tau} \in (N/N_{\tau})_{\mathbb{R}}$ inside the space $(N/N_{\tau})_{\mathbb{R}}$.* Let us be more precise about the meaning of the first part of this characterization. Choose an arbitrary linear projection ψ_{τ} of $N_{\mathbb{R}}$ onto the linear span $(N_{\tau})_{\mathbb{R}}$ of the cone τ . Then $(v_n)_{n \in \mathbb{N}}$ tends to infinity in the direction of the cone τ if and only if the sequence $(\psi_{\tau}(v_n))_{n \in \mathbb{N}}$ gets eventually out of any compact of $(N_{\tau})_{\mathbb{R}}$ and also enters eventually any fixed neighborhood of τ , also inside $(N_{\tau})_{\mathbb{R}}$.

EXAMPLE 4.8. Let $N = \mathbb{Z}^2$ and let σ be the convex polyhedral cone generated by the vectors $(1,0)$ and $(1,2)$ in \mathbb{R}^2 . Denote the 1-dimensional faces of σ by $\tau_1 = \langle (1,0) \rangle$ and $\tau_2 = \langle (1,2) \rangle$. The corresponding stratification of the disjoint union $L = \bigsqcup_{\tau \leq \sigma} (N/N_{\tau})_{\mathbb{R}}$ consists of the four pieces: $L_0 = \mathbb{R}^2$, $L_1 = \mathbb{R}^2/\mathbb{R} \cdot \tau_1$, $L_2 = \mathbb{R}^2/\mathbb{R} \cdot \tau_2$, and the point $L_{12} = \mathbb{R}^2/\mathbb{R} \cdot \sigma$. The cone σ and the stratification are schematically shown in Figure 3. Let the open set U be an open circle and $\delta = \tau_2$. Then the corresponding open subset of L is

$$U_{\tau_2} = (U + \tau_2) \sqcup \pi_2(U + \tau_2),$$

where $\pi_2 := \pi_{\tau_2}: \mathbb{R}^2 \rightarrow L_2$ is the canonical projection (see again Figure 3).

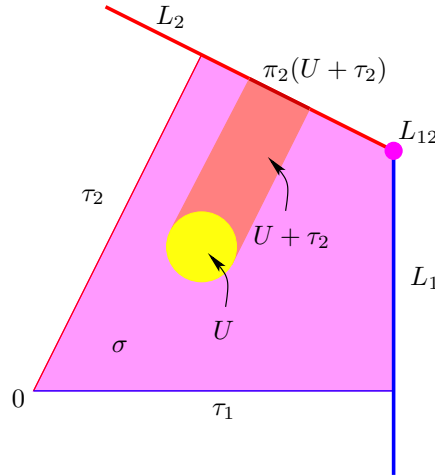


FIGURE 3. An affine linear variety of dimension 2

EXAMPLE 4.9. In order to indicate better the adjacencies of strata which appear by the construction of the affine linear variety associated to a pair (σ, N) , let us also represent a 3-dimensional situation. We consider a lattice N of rank 3 and inside $N_{\mathbb{R}}$ a strictly convex cone σ of dimension 3 having 4 edges, denoted τ_1, \dots, τ_4 . Denote also by τ_I the face of σ spanned by the subset I of $\{1, \dots, 4\}$, whenever we get indeed a face, and by $L_I := (N/N_{\tau_I})_{\mathbb{R}}$. In particular, $\tau_{1234} = \sigma$, therefore L_{1234} is a point. In Figure 4 we represented $L(\sigma, N)$, as well as the canonical projections $\pi_I(\sigma)$ of σ

to the strata at infinity L_I (where, as in the previous example, we denote $\pi_I := \pi_{\tau_I}$).

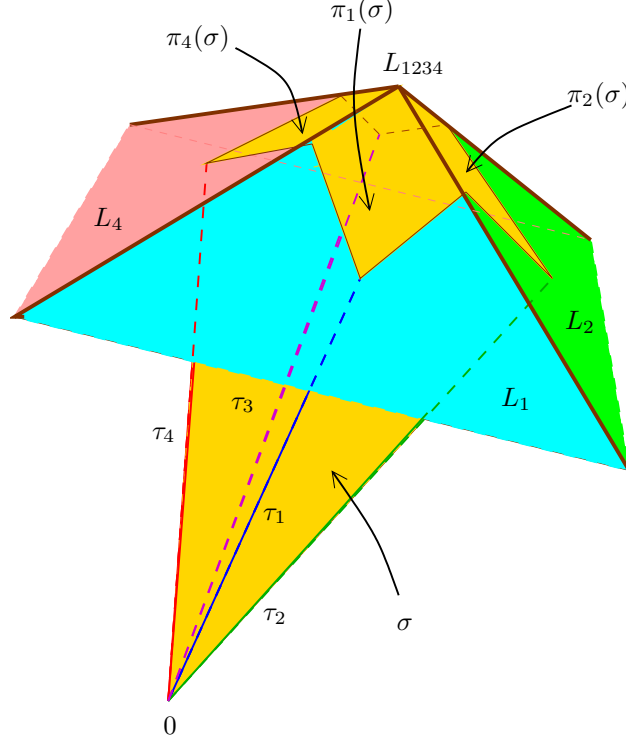


FIGURE 4. A 3-dimensional affine linear variety

PROPOSITION 4.10. *The sets of the form \overline{U}_δ , where U is an open subset of $N_\mathbb{R}$ and δ is a face of the cone σ , form a basis of open sets for a topology on $\bigsqcup_\tau (N/N_\tau)_\mathbb{R}$, where τ varies over the faces of the cone σ .*

PROOF. The proof is easy and left to the reader. \square

Note that any element γ of $\bigsqcup_\tau (N/N_\tau)_\mathbb{R}$ defines a semigroup homomorphism from $\check{\sigma} \cap M$ to $\overline{\mathbb{R}}$. Indeed, if $\gamma \in (N/N_\tau)_\mathbb{R}$, then γ defines a canonical homomorphism from $\tau^\perp \cap \check{\sigma} \cap M$ to \mathbb{R} , where τ^\perp is the subspace of $M_\mathbb{R}$ orthogonal to τ . Extend this homomorphism to $\check{\sigma} \cap M$ by setting $\gamma(m) = +\infty$ for all $m \in \check{\sigma} \cap M$, $m \notin \tau^\perp$. In this way we get a canonical map $\bigsqcup_{\tau \leq \sigma} (N/N_\tau)_\mathbb{R} \rightarrow L(\sigma, N)$. Now the stratification of $L(\sigma, N)$ may be described set-theoretically as follows:

LEMMA 4.11. *The canonical map $\bigsqcup_{\tau \leq \sigma} (N/N_\tau)_\mathbb{R} \rightarrow L(\sigma, N)$ is a homeomorphism.*

PROOF. Let us denote this canonical map by F . The injectivity of F follows directly from its construction. To show surjectivity, we consider a homomorphism $\gamma: \check{\sigma} \cap M \rightarrow \overline{\mathbb{R}}$. The set of $m \in \check{\sigma} \cap M$ satisfying $\gamma(m) = +\infty$ is a semigroup ideal of $\check{\sigma} \cap M$. Moreover, this ideal is prime, that is, if $\gamma(m_1 + m_2) = +\infty$, then $\gamma(m_1) = +\infty$ or $\gamma(m_2) = +\infty$. As we already said in Section 2, such an ideal can be only the complement in $\check{\sigma} \cap M$ of an

intersection $\tau^\perp \cap \check{\sigma}$ for some face τ of the cone σ . This clearly implies that γ is in the image of F .

Next, let us prove that the map F is continuous in both directions. Let U be an interval (a, b) , $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}$ or half-interval $(a, +\infty]$ of the extended real line $\overline{\mathbb{R}}$ and m an element of the semigroup $\check{\sigma} \cap M$. Suppose that $\delta^\perp \cap \check{\sigma}$ is the minimal face of $\check{\sigma}$ containing m , where δ is a face of σ . Consider the open subset $W = \{\gamma \in L(\sigma, N) \mid \gamma(m) \in U\}$ of $L(\sigma, N)$. If U is contained in \mathbb{R} , then $F^{-1}(W)$ consists of an open part $H \subseteq N_{\mathbb{R}}$ and some strata at infinity. Namely, $F^{-1}(W) = \bigsqcup_{\tau \leq \delta} \pi_\tau(H + \delta) = \overline{H}_\delta$, and this set is open in $\bigsqcup_{\tau \leq \sigma} (N/N_\tau)_{\mathbb{R}}$. Assume that $U = (a, +\infty]$. Let W_0 be the open subset of $L(\sigma, N)$ corresponding to the interval $U_0 = (a, +\infty)$. Then $F^{-1}(W_0)$ contains the open halfspace $H = \{\gamma \in N_{\mathbb{R}} \mid \gamma(m) > a\}$ of $N_{\mathbb{R}}$ and we can write:

$$F^{-1}(W) = \overline{H}_\sigma \subseteq \bigsqcup_{\tau \leq \sigma} (N/N_\tau)_{\mathbb{R}}.$$

This set is open by the definition of the topology on $\bigsqcup_{\tau \leq \sigma} (N/N_\tau)_{\mathbb{R}}$.

Let \overline{U}_δ be an open subset of $\bigsqcup_{\tau \leq \sigma} (N/N_\tau)_{\mathbb{R}}$ of the form (4.3). We show that the set $F(\overline{U}_\delta)$ is also open. First note that it suffices to assume that the open set U in $N_{\mathbb{R}}$ is an intersection of finite number of open half-spaces $U^+(m, a) = \{\gamma \mid \gamma(m) > a\}$ or $U^-(m, a) = \{\gamma \mid \gamma(m) < a\}$ for *some elements m of M contained in the interior of $\check{\sigma}$* (recall that the cone $\check{\sigma}$ has nonempty interior, thus it contains a basis for M) and for some $a \in \mathbb{R}$. Using this, we can further reduce to the case when U is actually one half-space of the form $U^+(m, a)$ or $U^-(m, a)$. In the first case:

$$F(\overline{U}_\delta) = F(\overline{U_\delta^+(m, a)}) =$$

$$\{\gamma \in L(\sigma, N) \mid \gamma(m) \in (a, +\infty]\} \setminus \bigcup_{m' \in \delta^\perp \cap \check{\sigma} \cap M} \{\gamma \in L(\sigma, N) \mid \gamma(m') = +\infty\}.$$

The set $\{\gamma \mid \gamma(m') = +\infty\}$ is closed in $L(\sigma, N)$, and the union that we subtract in the formula above is in fact finite. Therefore the set $F(\overline{U}_\delta)$ is open. In the second case:

$$F(\overline{U}_0) = F(\overline{U_0^-(m, a)}) = \{\gamma \in L(\sigma, N) \mid \gamma(m) < a\}$$

if $\delta = \{0\}$ and:

$$F(\overline{U}_\delta) = F(\overline{U_\delta^-(m, a)}) = L(\sigma, N) \setminus \bigcup_{m' \in \delta^\perp \cap \check{\sigma} \cap M} \{\gamma \in L(\sigma, N) \mid \gamma(m') = +\infty\}$$

if $\delta \neq \{0\}$. Again we conclude that $F(\overline{U}_\delta)$ is open. \square

Consider now a rational fan Δ . By analogy with $L(\sigma, N)$, define:

$$(4.4) \quad L(\Delta, N) = \mathcal{Z}_{\mathbb{R}}(\Delta, N) = \bigsqcup_{\tau \in \Delta} (N/N_\tau)_{\mathbb{R}}, \quad \overline{\Delta} = \bigsqcup_{\tau \in \Delta} \overline{\tau}.$$

The following result generalizes Proposition 4.10. Its proof is also left to the reader.

PROPOSITION 4.12. *The sets of the form \overline{U}_δ (as in (4.3)), when U is any open subset of $N_{\mathbb{R}}$ and δ is a cone of the fan Δ , form a basis of open sets for a topology on $L(\Delta, N)$.*

The following definition is to Definition 4.7 what the Definition of the toric variety associated to a fan is to that of an affine toric variety:

DEFINITION 4.13. Let N be a lattice and Δ a rational polyhedral fan in $N_{\mathbb{R}}$. The topological space $L(\Delta, N)$, endowed with the natural continuous action $N_{\mathbb{R}} \times L(\Delta, N) \rightarrow L(\Delta, N)$ extending the action of $N_{\mathbb{R}}$ on itself by translations is called **the linear variety** associated to the pair (N, Δ) . We say that the closure of Δ in $L(\Delta, N)$ is the **extended fan** $\overline{\Delta}$.

The fan Δ determines a fan $\text{Star}(\tau)$ in every stratum $(N/N_{\tau})_{\mathbb{R}}$ of the linear variety $L(\Delta, N)$. The cones of $\text{Star}(\tau)$ are the projections of those cones δ of Δ which contain the cone τ as a face. In the case of an affine linear variety $L(\sigma, N)$, the system of fans $\{\text{Star } \tau\}_{\tau \leq \sigma}$ has the following interpretation:

PROPOSITION 4.14. *The system of fans $\{\text{Star } \tau\}_{\tau \leq \sigma}$ gives a stratification of the subspace $\overline{\sigma} = \overline{(\sigma, N)}$ of the affine linear variety $L(\sigma, N)$:*

$$\overline{\sigma} = \bigsqcup_{\tau \leq \sigma} \text{Star}(\tau).$$

Notice that the subspace $\overline{\sigma}^{\circ}$ consists of the interior $\overset{\circ}{\sigma}$ of σ and all projections of $\overset{\circ}{\sigma}$ to the strata $(N/N_{\tau})_{\mathbb{R}}$ and it is indeed the interior of $\overline{\sigma}$ in the usual topological sense. Moreover, if τ is a face of σ , we define the *relative interior* $\overline{\tau}^{\circ}$ of τ inside $\overline{\sigma}$ to be the union of the usual relative interior $\overset{\circ}{\tau}$ of τ and all projections of $\overset{\circ}{\tau}$ to the strata $(N/N_{\rho})_{\mathbb{R}}$, where ρ is a face of τ .

We could have easily avoided using the lattice N and the rationality of the fan Δ for the construction of the linear variety $L(\Delta)$ (for instance, we could have defined $L(\sigma)$ as the set of semigroup morphisms from $\check{\sigma}$ to \overline{R} which are equivariant under the natural action of \mathbb{R}^*). In fact, these discrete data determine an additional *integral structure* on the linear variety, as a particular case of Remark 4.6, (2):

PROPOSITION 4.15. *If Δ is a rational polyhedral fan, then every stratum $(N/N_{\tau})_{\mathbb{R}}$ of the linear variety $L(\Delta, N)$ carries a lattice N/N_{τ} in such a way that the natural action of N on itself extends canonically to an action by addition of N on all the lattices N/N_{τ} , $\tau \in \Delta$.*

If we consider as starting data of the construction a fan of semigroups \mathcal{S} (see Definition 3.4), instead of simply the underlying fan of cones Δ , the supplementary structure induced on $L(\Delta, N)$ is the knowledge, for each stratum at infinity $(N/N_{\tau})_{\mathbb{R}}$, of the semigroup Γ_{τ} , seen as a special additive semigroup of linear functions on N .

5. Valuation spaces

In this section we present the material from valuation theory we need in the sequel. We will work only with valuations taking values in the extended real line $\overline{\mathbb{R}}$.

DEFINITION 5.1. Let R be a ring. A **real (ring) valuation** on R is a map $R \xrightarrow{w} \overline{\mathbb{R}}$ such that:

- (1) w is a morphism of semigroups from (R, \cdot) to $(\overline{\mathbb{R}}, +)$.
- (2) $w(0) = \infty$ and $w(1) = 0$.
- (3) $w(f + g) \geq \min\{w(f), w(g)\}$, for all $f, g \in R$.

The *trivial valuation* is the valuation on an integral domain which vanishes identically on $R \setminus \{0\}$.

In general, valuations take values in arbitrary totally ordered abelian groups extended by ∞ (see Zariski and Samuel's book [ZS], as well as Vaquié's introductory text [V]). As we will not use that level of generality, in the sequel by *valuation* we mean a real ring valuation.

The following is an immediate consequence of the definition:

LEMMA 5.2. *If $f, g \in R$, w is a valuation of R , and $w(f) \neq w(g)$, then $w(f + g) = \min\{w(f), w(g)\}$.*

DEFINITION 5.3. We denote by $\mathcal{V}(R)$ the set of valuations on R , endowed with the topology of pointwise convergence of maps from R to $\overline{\mathbb{R}}$. We call it **the valuation space of R** .

Recall that the topology of pointwise convergence is generated by the subsets of the form:

$$U_f = \{w_S \in \mathcal{V}(S) \mid w_S(f) \in U\}, \text{ for some } f \in S \text{ and some open } U \subset \overline{\mathbb{R}},$$

in the sense that its open sets are arbitrary unions of finite intersections of such sets (one says that these sets form a *subbasis* of the topology).

Any morphism of rings $\phi: S \rightarrow R$ induces, by composition, a function between the associated valuation spaces:

$$(5.1) \quad \mathcal{V}(R) \xrightarrow{\mathcal{V}(\phi)} \mathcal{V}(S), \quad \mathcal{V}(\phi)(v) = v \circ \phi.$$

PROPOSITION 5.4. *The function $\mathcal{V}(\phi)$ is continuous. Therefore, \mathcal{V} defines a contravariant functor from the category of rings to the category of topological spaces.*

PROOF. Consider an arbitrary subbasic open subset U_f of $\mathcal{V}(S)$, where $f \in S$ and U is open in $\overline{\mathbb{R}}$. Then its preimage:

$$(\mathcal{V}(\phi))^{-1}(U_f) = \{w_R \in \mathcal{V}(R) \mid w_R(\phi(f)) \in U\} = U_{\phi(f)}$$

is, by definition, an open subset of $\mathcal{V}(R)$. This shows that our map $\mathcal{V}(\phi)$ is continuous. \square

DEFINITION 5.5. We say that a valuation $w \in \mathcal{V}(R)$ is **centered in R** if $w(f) \geq 0$ for every $f \in R$. In this case the **center** of the valuation w is the prime ideal $\{f \in R \mid w(f) > 0\}$. The **home** of the valuation w is the prime ideal $\{f \in R \mid w(f) = \infty\}$.

The home of a valuation is characterized by the following lemma:

LEMMA 5.6. *A valuation $w \in \mathcal{V}(R)$ is the preimage of a valuation $\overline{w} \in \mathcal{V}(R/I)$ if and only if the ideal I is contained in the home of w .*

REMARK 5.7. Classically (see Zariski and Samuel's book [ZS]), the definition of valuations requires that non-zero elements take values in \mathbb{R} . Therefore, a valuation in the extended sense which we use in this paper is simply

obtained by pulling back a classical valuation from a quotient ring. We need such extended valuations, as they may appear as limits of classical ones. Since we work with valuations centered in a ring R , we do not need to add more points to the valuation space, as the next proposition shows.

PROPOSITION 5.8. *The space $\mathcal{V}_{\geq 0}(R)$ of valuations centered in R is compact.*

PROOF. By Tychonoff's theorem (see for instance [HY, Section 1-10]), the space $[0, \infty]^R$ is compact when endowed with the product topology. Therefore it is enough to prove that $\mathcal{V}_{\geq 0}(R)$ is closed inside $[0, \infty]^R$. But any function in $[0, \infty]^R$ which is a limit of valuations is also a valuation. Indeed, the axioms of Definition 5.1 depend on at most two elements of R , and those equalities or inequalities are preserved by the limit process. \square

REMARK 5.9. The previous proof is similar to Zariski's proof of the quasi-compactness of the Riemann-Zariski space S of a field extension K/k (see [ZS, Ch. VI, Sect. 17, Theorem 40]). By definition, the points of this space are the Krull-valuation rings of K containing k . The topology of S is obtained by taking as basis of open sets the subsets of valuation rings which contain a given finitely generated subring of K containing k . In order to get the announced quasi-compactness, Zariski embeds S in the space of maps from K to $\{-, 0, +\}$ (associating to each element of K the sign of its value). Then he uses Tychonoff's theorem for the space $\{-, 0, +\}^K$. A subtle point here is that in order to apply Tychonoff's theorem we must use the discrete topology on $\{-, 0, +\}$ (making it Hausdorff compact), but in order to get the correct topology on S we have to consider a weaker topology (having the full set and $\{0, +\}$ as basis of open sets), which is non-Hausdorff.

In the sequel, we will need to work with special subspaces of the valuation space of a ring. In addition to the spaces $\mathcal{V}_{\geq 0}(R)$ introduced in Proposition 5.8, the main types of subspaces we need are described in the next two definitions.

DEFINITION 5.10. Let $S \xrightarrow{\phi} R$ be a morphism of rings and let $w_S \in \mathcal{V}(S)$ be a fixed valuation. Denote by $\mathcal{V}_{(\phi, w_S)}(R) \subset \mathcal{V}(R)$ the set of valuations w_R on R such that $\mathcal{V}(\phi)(w_R) = w_S$. We call it **the valuation space of R relative to (ϕ, w_S)** . When S is a subring of R and $S \xrightarrow{\phi} R$ is the inclusion morphism, we also write: $\mathcal{V}_{(S, w_S)}(R) := \mathcal{V}_{(\phi, w_S)}(R)$, and we call it **the valuation space of R relative to (S, w_S)** .

REMARK 5.11. When S is a subfield of R and w_S is a valuation such that S is complete with respect to the associated norm e^{-w_S} , the relative valuation space $\mathcal{V}_{(S, w_S)}(R)$ is precisely the underlying topological space of the *Berkovich analytic space* associated to $\text{Spec} R$ (see [Berk]). One may consult Gubler [Gub 11] for relations between Berkovich analytification and tropicalization.

DEFINITION 5.12. Let \mathfrak{p} be a prime ideal of R . Denote by $\mathcal{V}(R, \mathfrak{p})$ the subspace of $\mathcal{V}(R)$ consisting of all valuations centered in R and whose center is precisely \mathfrak{p} . Call it **the valuation space of R relative to \mathfrak{p}** .

Any valuation $w \in \mathcal{V}(R, \mathfrak{p})$ extends to the localization $R_{\mathfrak{p}}$ by setting $w(f/g) = w(f) - w(g)$ for $f/g \in R_{\mathfrak{p}}$ and $f \in R, g \in R \setminus \mathfrak{p}$. Moreover, in this way we get a valuation from $\mathcal{V}(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}})$. Thus the spaces $\mathcal{V}(R, \mathfrak{p})$ and $\mathcal{V}(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}})$ are naturally homeomorphic.

As a particular case of the previous definition:

DEFINITION 5.13. If (R, \mathfrak{m}) is a local ring, we call $\mathcal{V}(R, \mathfrak{m})$ the **space of local valuations** of (R, \mathfrak{m}) .

We recall that a *local morphism* $(S, \mathfrak{n}) \xrightarrow{\phi} (R, \mathfrak{m})$ between local rings is a morphism of rings such that $\phi^{-1}(\mathfrak{m}) = \mathfrak{n}$. As a local analog of Proposition 5.4, we have:

PROPOSITION 5.14. *Let $(S, \mathfrak{n}) \xrightarrow{\phi} (R, \mathfrak{m})$ be a local morphism of local rings. Then the canonical map $\mathcal{V}(R, \mathfrak{m}) \xrightarrow{\mathcal{V}(\phi)} \mathcal{V}(S, \mathfrak{n})$ is continuous. Therefore, taking valuation spaces defines a contravariant functor from the category of local rings and local morphisms to the category of topological spaces.*

Given a valuation w on a ring R , we define the associated value of an ideal I :

$$(5.2) \quad w(I) := \inf\{w(f) \mid f \in I\}.$$

When R is Noetherian and w is nonnegative on R , any ideal is finitely generated, and the infimum is achieved, due to the following lemma:

LEMMA 5.15. *Suppose that the ideal I of the ring R is generated by f_1, \dots, f_r and let $w \in \mathcal{V}(R)$ be a valuation center in R . Then:*

$$w(I) = \min\{w(f_1), \dots, w(f_r)\}.$$

PROOF. This lemma follows directly from the definition of valuations and from nonnegativity of w on R . \square

Now, let (R, \mathfrak{m}) be a Noetherian local ring and let $(\hat{R}, \hat{\mathfrak{m}})$ be its completion with respect to \mathfrak{m} .

LEMMA 5.16. *Let $w \in \mathcal{V}(R, \mathfrak{m})$. Then w is continuous with respect to the \mathfrak{m} -adic topology. It may therefore be extended by continuity to a valuation in $\mathcal{V}(\hat{R}, \hat{\mathfrak{m}})$.*

PROOF. Since the ring R is Noetherian, Krull's theorem implies that it is separated in its \mathfrak{m} -adic topology, that is, $\bigcap_{n \in \mathbb{N}} \mathfrak{m}^n = 0$ (see [AM, Corollary 10.18]). Using Lemma 5.15, we see that:

$$m_0 := w(\mathfrak{m}) \in (0, \infty].$$

The same lemma implies that $w(\mathfrak{m}^n) = n \cdot m_0$, for all $n \in \mathbb{N}$.

Consider any $f \in R$. By the definition of the \mathfrak{m} -adic topology, the sets $(f + \mathfrak{m}^n)_{n \in \mathbb{N}}$ form a basis of neighborhoods of f . We consider now two cases, according to the value of $w(f)$.

- First, suppose that $w(f) \neq \infty$. Then, there exists $n_0 \in \mathbb{N}$ such that $n \cdot m_0 > w(f)$ for any $n \geq n_0$. For such a value of n , consider any $g \in f + \mathfrak{m}^n$, and we write $g = f + \mu$, with $\mu \in \mathfrak{m}^n$. Therefore $w(f) < w(\mu)$, which by Lemma 5.2 implies that $w(g) = w(f)$. Thus, w is constant in the neighborhood $f + \mathfrak{m}^n$ of f , and so it is continuous at f .

• Secondly, *suppose that $w(f) = \infty$* . We split this situation into two subcases:

- (1) *If $m_0 = \infty$, then we see that $w(g) = \infty$ for any $g \in f + \mathfrak{m}$, which implies again that w is continuous in a neighborhood of f .*
- (2) *If $m_0 \in (0, \infty)$, then we see that $w(g) \geq nm_0$ for all $g \in f + \mathfrak{m}^n$, which shows again that w is continuous at f .*

□

As a consequence, we can canonically identify the valuation space of a local Noetherian ring with the one of its completion.

COROLLARY 5.17. *The inclusion $(R, \mathfrak{m}) \xrightarrow{i} (\hat{R}, \hat{\mathfrak{m}})$ induces an isomorphism of local valuation spaces: $\mathcal{V}(\hat{R}, \hat{\mathfrak{m}}) \xrightarrow{\mathcal{V}(i)} \mathcal{V}(R, \mathfrak{m})$.*

PROOF. By the previous lemma applied to $(\hat{R}, \hat{\mathfrak{m}})$, any $\hat{w} \in \mathcal{V}(\hat{R}, \hat{\mathfrak{m}})$ is continuous for the \mathfrak{m} -adic topology. Therefore it is determined by its restriction to R , which proves the injectivity of $\mathcal{V}(i)$. The surjectivity follows from Lemma 5.16. □

The next lemma shows that we can reduce the study of the valuation space of an affine scheme to those of the irreducible components of the associated reduced scheme.

LEMMA 5.18. *Let R be a ring and $R \xrightarrow{\rho} R_{\text{red}}$ be its reduction morphism (that is, the morphism of quotient by its nilradical). Then, the map $\mathcal{V}(\rho)$ induced by ρ is a homeomorphism of $\mathcal{V}(R)$ and $\mathcal{V}(R_{\text{red}})$. If R is reduced and the $(\mathfrak{p}_i)_{i \in I}$ are the prime ideals of the primary decomposition of the zero ideal, then $\mathcal{V}(R) = \bigcup_{i \in I} \mathcal{V}(R/\mathfrak{p}_i)$, that is, the valuation space of $\text{Spec } R$ is the union of the valuation spaces of its irreducible components. The same holds for the space $\mathcal{V}(R, \mathfrak{p})$ of valuations relative to a prime ideal \mathfrak{p} of R .*

PROOF. The result follows from Lemma 5.6 and the fact that if w is a valuation of R , then the home of w contains necessarily at least one of the ideals \mathfrak{p}_i of the primary decomposition of $\{0\}$. □

6. An affine theory of tropicalization

In this section we describe our proposed framework for a theory of tropicalization which both generalizes the existing one of tropicalization of subvarieties of tori and allows in particular to tropicalize (even formal) germs of subvarieties of toric varieties. We stress also the functorial properties of our notion of tropicalization. The qualificative “*affine*” in the title of this section is explained in Remark 6.13. In Section 14 we describe a more general framework for functorial tropicalization.

In the sequel, $(\Gamma, +)$ denotes an arbitrary *affine semigroup* and $(R, +, \cdot)$ a commutative ring. Consider a *morphism of semigroups*:

$$(\Gamma, +) \xrightarrow{\gamma} (R, \cdot).$$

This is the same as giving a morphism of rings $\mathbb{Z}[\Gamma] \xrightarrow{\gamma} R$, and, thus, a morphism of schemes $\text{Spec}(R) \xrightarrow{u} \text{Spec}(\mathbb{Z}[\Gamma])$.

If $w \in \mathcal{V}(R)$, we have $w \circ \gamma \in \text{Hom}_{S_g}(\Gamma, \overline{\mathbb{R}})$. By formula (4.1), we see that $w \circ \gamma \in L(\sigma(\Gamma), N(\Gamma))$. We can define:

$$(6.1) \quad \mathcal{V}(R) \xrightarrow{\Phi_\gamma} L(\sigma(\Gamma), N(\Gamma)).$$

It is a routine exercise to check:

LEMMA 6.1. *The map Φ_γ is continuous with respect to the topologies of pointwise convergence on $\mathcal{V}(R)$ and $L(\sigma(\Gamma), N(\Gamma))$.*

In the next definition, we allow \mathcal{W} to be *any* subset of $\mathcal{V}(R)$. In the sequel we will be particularly interested in these subsets of valuation spaces relative to valuations defined on subrings (see Definition 5.10) or to ideals (see Definition 5.12).

DEFINITION 6.2. Let \mathcal{W} be a subset of the valuation space $\mathcal{V}(R)$. The closure in $L(\sigma(\Gamma), N(\Gamma))$ of the image $\Phi_\gamma(\mathcal{W})$ is called **the (global) tropicalization of \mathcal{W} with respect to the semigroup morphism γ** , and it is denoted by $\text{Trop}(\mathcal{W}, \gamma)$ or $\text{Trop}(\mathcal{W}, u)$.

REMARK 6.3. One of the definitions of tropicalization of subvarieties of tori proposed by [EKL] corresponds to the case where Γ is the lattice of exponents of monomials of the torus $\text{Spec}(K[\Gamma])$, $R = K[\Gamma]/I$ for an ideal I of $K[\Gamma]$, γ is the composition $\Gamma \hookrightarrow K[\Gamma] \rightarrow K[\Gamma]/I$, and $\mathcal{W} = \mathcal{V}_{K,v}(R)$ is the set of valuations on R extending a valuation v of the field K .

REMARK 6.4. Our definition is indeed more general than the one explained in the previous remark. More precisely, if Γ is an affine semigroup, and $\gamma: \Gamma \rightarrow (R, \cdot)$ is a morphism of semigroups, then γ does not extend in general to a morphism from the associated lattice $M(\Gamma)$ of Γ . In fact, such an extension exists if and only if the image of γ is contained in the group of units of (R, \cdot) .

REMARK 6.5. When K is a field and I is an ideal of the ring $K[\Gamma]$, we set:

$$\text{Trop}(I) := \text{Trop}(\mathcal{V}(R), \gamma),$$

where $R := K[\Gamma]/I$ and $\gamma: \Gamma \rightarrow R$ is the morphism of semigroups induced by the quotient map $K[\Gamma] \rightarrow K[\Gamma]/I$. When Γ is saturated, this agrees with the notion of tropicalization of a subvariety of a normal affine toric variety introduced by Payne [Pay08]. In fact, these tropicalizations may be glued to produce a tropicalization of an arbitrary subscheme of a general (not necessarily normal) toric variety. In this case, if the toric variety is clear from the context and X is a subscheme of it, we denote this tropicalization simply by $\text{Trop}(X)$.

Next, we define the notion of *local tropicalization*. Denote by σ the cone $\sigma(\Gamma) \subset N(\Gamma)_{\mathbb{R}}$. Let (R, \mathfrak{m}) be a local ring and $\gamma: \Gamma \rightarrow R$ be a *local* morphism of semigroups, i.e., $\gamma^{-1}(\mathfrak{m}) = \Gamma^+$. Recall from Definition 5.13 that by a *local valuation* of R we mean a valuations nonnegative on R and positive on \mathfrak{m} , that is, an element of the space $\mathcal{V}(R, \mathfrak{m})$. Note that, by (4.2), the map Φ_γ considered above sends the space $\mathcal{V}(R, \mathfrak{m})$ into the extended cone $\overline{\sigma}$ (see Definition 4.7).

DEFINITION 6.6. The **local positive tropicalization** of γ , denoted $\text{Trop}_{>0}(\mathcal{V}(R, \mathfrak{m}), \gamma)$ or simply $\text{Trop}_{>0}(\gamma)$, is the closure of the image of the map $\nu(R, \mathfrak{m}) \xrightarrow{\Phi_\gamma} L(\sigma, N)$ in the relative interior $\bar{\sigma}^\circ$ of the space $\bar{\sigma}$.

Notice that in this definition we only consider those valuations of R which have as center the closed point of $\text{Spec } R$. Instead, if we only require that the valuations have a center on $\text{Spec } R$, possibly smaller than \mathfrak{m} , we get another version of local tropicalization, which will also be important in the sequel:

DEFINITION 6.7. The **local nonnegative tropicalization** of γ , denoted $\text{Trop}_{\geq 0}(\gamma)$, is the image in the extended cone $\bar{\sigma}$ of the map Φ_γ applied to all valuations of R having a center on $\text{Spec } R$, that is, all nonnegative valuations of R .

The following proposition states direct consequences of the definitions of the two kinds of local tropicalizations:

PROPOSITION 6.8. *Let Γ be an arbitrary affine semigroup and $\sigma = \sigma(\Gamma)$.*

- (i) *The local nonnegative tropicalization is a closed subset of $\bar{\sigma}$.*
- (ii) *If the set $\gamma(\Gamma^+) \subseteq \mathfrak{m}$, where Γ^+ is the maximal ideal of Γ , generates \mathfrak{m} (as an ideal of the ring R), then the image of the map $\Phi_\gamma: \nu(R, \mathfrak{m}) \rightarrow L(\sigma, N)$ coincides with $\text{Trop}_{\geq 0}(\gamma) \cap \bar{\sigma}^\circ$. In particular, this image is closed in $\bar{\sigma}^\circ$ and $\text{Trop}_{>0}(\gamma) = \text{Trop}_{\geq 0}(\gamma) \cap \bar{\sigma}^\circ$.*

PROOF. Statement (i) follows from Proposition 5.8 and Lemma 6.1. Statement (ii) follows from (i) and the definition of positive tropicalization. In general, the question of closedness of the image of Φ_γ is subtler and connected to the problem of extension of valuations, see Section 7. \square

REMARK 6.9. (Local analog of Remark 6.5). When K is a field, Γ is an affine pointed semigroup and I is an ideal of the ring $K[[\Gamma]]$ of formal power series with exponents in Γ (discussed more carefully in Section 8), we denote $\text{Trop}_{>0}(I) := \text{Trop}_{>0}(\gamma)$ and $\text{Trop}_{\geq 0}(I) := \text{Trop}_{\geq 0}(\gamma)$.

DEFINITION 6.10. Let (S, \mathfrak{n}) be a local subring of (R, \mathfrak{m}) , endowed with a valuation $w_S \in \mathcal{V}(S, \mathfrak{n})$. Denote by $\mathcal{V}_{(S, w_S)}(R, \mathfrak{m})$ the set of valuations in $\mathcal{V}(R, \mathfrak{m})$ which extend w_S , called **the valuation space of (R, \mathfrak{m}) relative to $((S, \mathfrak{n}), w_S)$** . Then $\text{Trop}_{>0}(\mathcal{V}_{(S, w_S)}(R, \mathfrak{m}), \gamma)$ is called **the local positive tropicalization of the semigroup morphism γ relative to $((S, \mathfrak{n}), w_S)$** . We denote it by $\text{Trop}_{>0}(R, (S, w_S), \gamma)$.

Let us now discuss the functorial properties of our definition of tropicalization (both local and global).

DEFINITION 6.11. Consider two semigroup morphisms $\Gamma_i \xrightarrow{\gamma_i} (R_i, \cdot)$, for $i = 1, 2$. A **morphism from γ_1 to γ_2** is a pair of maps:

$$(\phi_H \in \text{Hom}_{Rg}(R_1, R_2); \lambda_H \in \text{Hom}_{Sg}(\Gamma_1, \Gamma_2))$$

making the following diagram commutative:

$$\begin{array}{ccc} R_1 & \xrightarrow{\phi_H} & R_2 \\ \gamma_1 \uparrow & & \uparrow \gamma_2 \\ \Gamma_1 & \xrightarrow{\lambda_H} & \Gamma_2 \end{array}$$

We denote by **SgRg** the category defined in this way, and by **SgRgVal** the category whose objects are pairs $(\Gamma \xrightarrow{\gamma} (R, \cdot), \mathcal{W} \subset \mathcal{V}(R))$ and whose morphisms are the morphisms of the category **SgRg** which respect the chosen subsets of the valuation spaces (that is, which send one into the other).

PROPOSITION 6.12. *Let $(\Gamma_i \xrightarrow{\gamma_i} (R_i, \cdot), \mathcal{W}_i)$, for $i = 1, 2$ be two objects of the category **SgRgVal** and H a morphism from $(\gamma_1, \mathcal{W}_1)$ to $(\gamma_2, \mathcal{W}_2)$. Then H induces a functorial linear map:*

$$\mathrm{Trop}(\mathcal{W}_2, \gamma_2) \xrightarrow{\mathrm{Trop}(H)} \mathrm{Trop}(\mathcal{W}_1, \gamma_1),$$

Moreover, positive tropicalizations are preserved by $\mathrm{Trop}(H)$.

REMARK 6.13. We call the theory developed in this section “affine”, because we think about the category **SgRgVal** as the analog of affine schemes. A next step, which we do not develop in this paper (for some follow up on this matter, see Sections 13 and 14), would be to glue objects of this “affine” category into non-affine objects which may again be tropicalized.

7. Extensions of valuations

In this section we address the problem of extending of a valuation from a ring to a bigger ring, in a generality suitable for our purposes. As an application, we show that under convenient hypothesis, the real part of the local tropicalization is necessarily non-empty (see Lemma 7.4), and that tropicalization is unchanged by passage to the normalization (see Corollary 7.6).

The following extension principle plays an important role in [BG]: if $K \subseteq L$ is a field extension and v is a real valuation of K , then v can always be extended to a *real* valuation w of L , that is there is a real valuation w of L such that w restricted to K coincides with v . We now give a local version of this extension principle. Let (R, \mathfrak{m}) and (S, \mathfrak{n}) be two local rings such that $R \subseteq S$, $\mathfrak{n} \cap R = \mathfrak{m}$, and let v be a local real ring valuation of the ring R , i.e., v is nonnegative on R and positive on the maximal ideal \mathfrak{m} . We address the following question: *Does there exist a local real valuation w of the ring S such that w restricted to R coincides with v ?*

As a first approach, we may assume that the given valuation v is only nonnegative on \mathfrak{m} , and ask whether there exists an extension w nonnegative on \mathfrak{n} . Geometrically, we consider only valuations centered at the maximal ideals of our local rings, or, if v and w are only nonnegative on maximal ideals, such that their centers (thought geometrically as irreducible subschemes) contain the special points of $\mathrm{Spec} R$ and $\mathrm{Spec} S$ respectively. The answer to this question is not always positive, as shown by the following simple example.

EXAMPLE 7.1. Let $R = K[[x, y]]$, $S = K[[s, t]]$ be two copies of the ring of formal power series in two variables over a field K , and assume that the inclusion of R into S is given by the map $x \mapsto s$, $y \mapsto st$ (this corresponds to the blow up of a point in a plane). Let v be a monomial valuation on R , trivial on K , and determined by $v(x) = 1$, $v(y) = 1$. Then v cannot be lifted to a local valuation w of S because w must take value 0 on t . If we set $v(x) = 2$, $v(y) = 1$, then it is impossible to find a nonnegative extension w , because $w(t)$ must be equal to -1 .

Next, we derive some sufficient conditions for the extension principle to hold.

THEOREM 7.2. *Let (R, \mathfrak{m}) and (S, \mathfrak{n}) be two local rings, $R \subseteq S$, $\mathfrak{n} \cap R = \mathfrak{m}$. Let v be a real nonnegative ring valuation on R , and assume that one of the following conditions holds:*

- a) *S is an integral extension of R (e.g. S is a finite R -module);*
- b) *R and S are Noetherian, complete with respect to the \mathfrak{m} -adic and \mathfrak{n} -adic topologies, and (i) S is flat over R , (ii) the residue fields R/\mathfrak{m} and S/\mathfrak{n} are naturally isomorphic, and (iii) the fiber of the scheme $\text{Spec}(S)$ over the maximal ideal \mathfrak{m} of $\text{Spec}(R)$ is reduced and irreducible, i. e., the ideal $\mathfrak{m}S$ is prime in S .*

Then, there is a real nonnegative valuation w of the ring S such that w restricted to R coincides with v . If, moreover, v is local (that is, positive on \mathfrak{m}), then w can also be chosen to be local. In fact, in case a) every valuation w extending v is nonnegative, and local if v is local.

PROOF. First we prove the sufficiency of condition a). Let \mathfrak{p} be the home of the valuation v . By basic properties of integral extensions (see, e. g., [AM, Chapter 5]) there exists a prime ideal \mathfrak{q} of S such that $\mathfrak{q} \cap R = \mathfrak{p}$. Then, S/\mathfrak{q} is an integral extension of R/\mathfrak{p} . By Lemma 5.6, v defines a valuation v' of the ring R/\mathfrak{p} , and it suffices to extend the valuation v' to S/\mathfrak{q} . This shows that from the beginning we can assume that S and R are local domains and the home of v is $\{0\}$. Let $K(R)$ and $K(S)$ denote the fields of fractions of R and S respectively, so $K(R) \subseteq K(S)$. The valuation v can be defined on $K(R)$ by the rule $v(a/b) = v(a) - v(b)$. As we have already mentioned at the beginning of this section, valuations from fields can always be extended, so let w be any real valuation of the field $K(S)$ extending v from $K(R)$. Since v is nonnegative on R , the valuation ring S_w of w contains R . On the other hand, the integral closure of R in the field $K(S)$ is the intersection of all valuation rings of $K(S)$ containing R ([AM, Corollary 5.22]), thus S is contained in S_w and w is nonnegative on S .

Now assume that the valuation v is local, and let w be any extension of it. We have just seen that w is nonnegative on S . Consider the set I of elements x of S which satisfy an integral dependence relation:

$$f(x) = x^n + r_1 x^{n-1} + \dots + r_n = 0$$

with $r_1, \dots, r_n \in \mathfrak{m}$ and $n \in \mathbb{N}$. Fix such an x , and let $s \in S$. The element s also satisfies an integral dependence relation:

$$g(s) = s^m + a_1 s^{m-1} + \dots + a_m = 0,$$

where $a_1, \dots, a_m \in R$. Let $s_1 = s, s_2, \dots, s_m$ be all the roots of g in the field $\overline{K(S)}$. Consider the polynomial:

$$F(X) = (s_1 \cdots s_m)^n \prod_{i=1}^m f\left(\frac{X}{s_i}\right).$$

This is a monic polynomial in the variable X and, moreover, its coefficients are symmetric polynomials in s_1, \dots, s_m with coefficients in \mathfrak{m} . It follows that F has coefficients in \mathfrak{m} , and, since $F(sx) = 0, sx \in I$. Consider one more element $y \in I$. Let:

$$h(y) = y^d + t_1 y^{d-1} + \dots + t_d = 0,$$

where $t_1, \dots, t_d \in \mathfrak{m}$, be the corresponding integral dependence relation. Let $y_1 = y, \dots, y_d$ be the roots of h in $\overline{K(S)}$. Applying the same argument to the polynomial:

$$H(X) = \prod_{i=1}^d h(X - y_i),$$

we show that $x + y \in I$. Thus I is an ideal of the ring S . Clearly $I \cap R = \mathfrak{m}$. It follows from [AM, Proposition 4.2 and Corollary 5.8] that I is \mathfrak{n} -primary and if s is any element of the maximal ideal \mathfrak{n} of S , then $s^k \in I$ for some k (in fact, we thus have $I = \mathfrak{n}$). But any $x \in I$ should satisfy $w(x) > 0$, because otherwise we would have:

$$w(x^n + r_1 x^{n-1} + \dots + r_n) = \min\{w(x^n), w(r_1 x^{n-1}), \dots, w(r_n)\} = 0.$$

It follows that $w(s) > 0$. As $s \in \mathfrak{n}$ is arbitrary, we see that w is also local.

Now we prove the *sufficiency* of condition b). By Theorem 7.3, we can find analytically independent elements $x_1, \dots, x_k \in S$ over R such that S is a finite module over $R[[x_1, \dots, x_k]]$. First we have to extend the valuation v to the intermediate ring $R' = R[[x_1, \dots, x_k]]$. For this we choose any positive real values $w'(x_1), \dots, w'(x_k)$ and for any $f = \sum_m a_m x^m \in R'$, $x^m = x_1^{m_1} \cdots x_k^{m_k}$, $a_m \in R$, we define:

$$w'(f) = \min_m \{v(a_m) + m_1 w'(x_1) + \dots + m_k w'(x_k)\}.$$

We can easily check that this defines a nonnegative (local if v is local) valuation w' on the ring R' . Then by a) we can extend w' from R' to S . This concludes the proof. \square

The proof of the following result was communicated to us by Mark Spivakovsky.

THEOREM 7.3. *Let R and S be local rings satisfying the assumptions of condition b) of Theorem 7.2. Then, there exists a finite number of elements x_1, \dots, x_k of S which are analytically independent over R , such that the extension $R \subseteq S$ factorizes as:*

$$R \subseteq R[[x_1, \dots, x_k]] \subseteq S,$$

and S is a finite module over $R[[x_1, \dots, x_k]]$.

PROOF. The rings R and S are local and Noetherian, hence they both have finite Krull dimension. Since S is flat over R , $\dim S - \dim R = \dim S/\mathfrak{m}S$ (see [Mats, Theorem 19 (2), p. 79]). We denote this number by k . Let x_1, \dots, x_k be elements of $\mathfrak{n} \setminus \mathfrak{m}$ whose images in $\mathfrak{n}(S/\mathfrak{m}S)$ form a system of parameters. The fact that x_1, \dots, x_k are analytically independent over R follows from the local criterion of flatness ([Mats, Theorem 49 (4), p. 147]), which says that S is R -flat if and only if $S/\mathfrak{m}S$ is R/\mathfrak{m} -flat and the canonical maps:

$$\gamma_n: (\mathfrak{m}^n/\mathfrak{m}^{n+1}) \bigotimes_{R/\mathfrak{m}R} (S/\mathfrak{m}S) \rightarrow \mathfrak{m}^n S/\mathfrak{m}^{n+1} S$$

are isomorphisms. Indeed, suppose that there is an analytic dependence relation:

$$(7.1) \quad \sum_m a_m x^m = 0,$$

where $m = (m_1, \dots, m_k)$, $x^m = x_1^{m_1} x_2^{m_2} \dots x_k^{m_k}$, $a_m \in R$. Denote by n the smallest nonnegative integer such that $a_m \notin \mathfrak{m}^{n+1}$ for some m . Then relation (7.1) gives rise to a relation of the form:

$$\sum_{i=1}^t b_i f_i = 0$$

with $b_i \in \mathfrak{m}^n/\mathfrak{m}^{n+1}$, $f_i \in R[[x_1, \dots, x_k]]/\mathfrak{m}R[[x_1, \dots, x_k]]$, which holds in $\mathfrak{m}^n S/\mathfrak{m}^{n+1} S$. Thus, the element:

$$\sum_{i=1}^t b_i \otimes f_i \in (\mathfrak{m}^n/\mathfrak{m}^{n+1}) \bigotimes_{R/\mathfrak{m}R} (S/\mathfrak{m}S),$$

where b_i are as above and f_i are now considered as elements of $S/\mathfrak{m}S$, is a nonzero element of the kernel of the canonical map γ_n , but this contradicts the quoted criterion of flatness.

Now we prove that the ring S is finite over $R' = R[[x_1, \dots, x_k]]$. Note that R' is also a Noetherian complete local ring with maximal ideal \mathfrak{m}' generated by \mathfrak{m} and x_1, \dots, x_k . Consider the extension $R/\mathfrak{m} \subseteq S/\mathfrak{m}S$ of complete local rings. Note that $S/\mathfrak{m}S$ is a domain and its residue field is isomorphic to R/\mathfrak{m} , this follows from assumptions b) (ii) and (iii) of Theorem 7.2. Then we can apply [Nag, Corollary 31.6, p. 109], which states that if x_1, \dots, x_k is a system of local parameters for $S/\mathfrak{m}S$, then S is finite over $(R/\mathfrak{m})[[x_1, \dots, x_k]]$. But then $S/\mathfrak{m}S$ is also finite over R' . By [Nag, Theorem 30.6, p. 105] we conclude that S is a finite module over R' . This finishes the proof of Theorem 7.3. \square

Theorem 7.2 may be expressed geometrically in the following way: *Any (flat) deformation of an algebroid germ over another such germ may be obtained as a finite (ramified) covering of the product of the base germ with a smooth algebroid variety.*

Corollary 5.17 implies that when working with local tropicalization we can always pass to complete rings. Note also that the positive tropicalization is never empty, since any local ring possesses the trivial local valuation v ,

where $v(r) = 0$ if $r \notin \mathfrak{m}$ and $v(r) = \infty$ if $r \in \mathfrak{m}$. Under rather general assumptions on R and some natural restrictions on γ the real part of the positive tropicalization is also nonempty.

LEMMA 7.4. *Assume that (R, \mathfrak{m}) is a complete local Noetherian domain, Γ is an affine semigroup, and $\gamma: \Gamma \rightarrow R$ is any local semigroup morphism such that no element of Γ goes to 0. Then $\text{Trop}_{>0}(\gamma) \cap \sigma \neq \emptyset$.*

PROOF. By the Cohen structure theorem for complete local rings ([Nag, Corollary 31.6, p. 109]) we know that R is a finite module over a subring of the form $J[[x_1, \dots, x_k]]$, where J is either a field or a discrete valuation ring. In the first case, we choose v to be the trivial valuation on J . In the second case, let v be the unique discrete valuation of J such that $v(p) = 1$ for the generator p of the maximal ideal of J . Then, we can extend v to $J[[x_1, \dots, x_k]]$ by assigning any positive values to x_1, \dots, x_k . By Theorem 7.2 a), this valuation can be extended to a valuation w of R with home $\{0\}$. This w is a point of $\text{Trop}_{>0} \gamma$ contained in σ . \square

In the next application of Theorem 7.2 we show, essentially, that the tropicalization does not change if we pass to the normalization.

LEMMA 7.5. *Let R be an integral domain, $\gamma: \Gamma \rightarrow R \setminus \{0\}$ a morphism from an affine semigroup Γ , and S the integral closure of R in its field of fractions $Q(R)$. Then there exists a unique extension $\bar{\gamma}: \text{Sat}(\Gamma) \rightarrow S$ of γ , and:*

$$\text{Trop}(\gamma, \mathcal{V}(R)) = \text{Trop}(\bar{\gamma}, \mathcal{V}(S)),$$

where $\mathcal{V}(R)$ and $\mathcal{V}(S)$ denote the spaces of all valuations of R and of S respectively. If R and S are both local, then also $\text{Trop}_{\geq 0}(\gamma) = \text{Trop}_{\geq 0}(\bar{\gamma})$ and $\text{Trop}_{>0}(\gamma) = \text{Trop}_{>0}(\bar{\gamma})$.

PROOF. Since none of the elements of Γ goes to 0, the morphism γ extends uniquely to a homomorphism from $M(\Gamma)$ to $Q(R)$. But the images of elements of $\text{Sat}(\Gamma)$ are obviously integral over R , thus they belong to S . Any valuation of R extends to a valuation of S , and its values on $\text{Sat}(\Gamma)$ are uniquely determined by its values on Γ . Moreover, by Theorem 7.2, any nonnegative valuation of R extends to a nonnegative valuation of S . This implies all the equalities of tropicalizations. \square

COROLLARY 7.6. *Under the conditions of Lemma 7.5, assume that I is an ideal of R and I is disjoint from the semigroup Γ . Let $p: R \rightarrow R/I$ and $q: S \rightarrow S/SI$ be the canonical projections. Then:*

$$\text{Trop}(p \circ \gamma, \mathcal{V}(R/I)) = \text{Trop}(q \circ \bar{\gamma}, \mathcal{V}(S/SI)).$$

If R and S are both local, then $\text{Trop}_{\geq 0}(p \circ \gamma) = \text{Trop}_{\geq 0}(q \circ \bar{\gamma})$, and similarly for the positive tropicalization.

PROOF. It suffices to show that if v is a valuation of R such that the home of v contains I , then the home of any extension \bar{v} of v to S contains SI . But indeed, if $f, g \in R$ and $v(f) = v(g) = \infty$, then for any $a, b \in S$ we have $\bar{v}(af + bg) = \infty$. \square

REMARK 7.7. In view of the previous results, the reader could wonder why we made the effort to develop a general framework of tropicalization for non-necessarily saturated affine semigroups. We see two reasons for this:

- Even if we take a morphism between two normal affine toric varieties (corresponding therefore to saturated affine semigroups), the closure of its image is again toric, but it may be non-normal. An example is given by the map from \mathbb{C} to \mathbb{C}^2 defined by $t \rightarrow (t^2, t^3)$, which is a parametrization of the cuspidal plane cubic. As another example, consider the parametrization $\phi : \mathbb{C}^2 \rightarrow \mathbb{C}^3$ defined by $(s, t) \rightarrow (x, y, z) = (st, s, t^2)$ of the Whitney umbrella, defined by the equation $x^2 - y^2z = 0$ in \mathbb{C}^3 .
- A morphism between two affine toric varieties does not necessarily lift to a morphism between their normalizations. For instance, consider the embedding of the singular locus of the Whitney umbrella W (defined in the previous example) into W . This singular locus S is the z -axis, therefore the embedding $S \hookrightarrow W$ may be described as a restriction of the toric map $\mathbb{C} \rightarrow \mathbb{C}^3$ given by $u \rightarrow (x, y, z) = (0, 0, u)$. The morphism ϕ of the previous example is a normalization map of W . The restriction of ϕ to $\phi^{-1}(S)$ is a double covering $\mathbb{C} \rightarrow \mathbb{C}$, therefore the map $S \rightarrow W$ does not lift to a map from S (equal to its own normalization) to the normalization \mathbb{C}^2 of W .

8. The formal toric rings $K[[\Gamma]]$

In this section we explain basic properties of rings of formal power series over K with exponents in pointed affine semigroups Γ . We call them “*formal toric rings*”, as they are the completions of the rings of the affine toric varieties $\text{Spec } K[\Gamma]$ at the unique closed orbit. In fact, till Corollary 8.6 we deal with arbitrary affine semigroups (satisfying perhaps a technical condition, as in Lemma 8.5). Then we restrict to the pointed ones.

Let K be a field and Γ an affine semigroup (see Definition 2.4). Whenever we want to use multiplicative notation for the elements of the semigroup Γ (which happens when we look at them as *monomials*), we write χ^m instead of m . We will say that m is the *exponent* of the *monomial* χ^m .

Recall that Γ^* denotes the subgroup of invertible elements of Γ . They are related by the short semigroup exact sequence:

$$(8.1) \quad 0 \longrightarrow \Gamma^* \longrightarrow \Gamma \xrightarrow{p} \Gamma' \longrightarrow 0,$$

where p is the quotient map of the semigroup Γ by the subgroup Γ^* .

By Proposition 2.13, if Γ is saturated, then Γ' is an affine semigroup. Since its subgroup of units is trivial, Γ' is pointed.

LEMMA 8.1. *Suppose that the affine semigroup Γ is saturated. Then the morphism p admits a section and any section induces a splitting of (8.1).*

PROOF. As explained in the proof of Proposition 2.13, we have the following commutative diagram in which the horizontal lines are short exact

sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma^* & \longrightarrow & \Gamma & \xrightarrow{p} & \Gamma' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Gamma^* & \longrightarrow & M(\Gamma) & \xrightarrow{M(p)} & M(\Gamma') \longrightarrow 0 \end{array}$$

As $M(\Gamma')$ is free, the surjective morphism of groups $M(p)$ admits a section α . This shows that the second exact sequence splits. Let us restrict α to Γ' . We see immediately that $\alpha(\Gamma') \subset \Gamma$, which shows that α is also a section of p . Define then the semigroup morphism $\Gamma \xrightarrow{\Phi} \Gamma^* \times \Gamma'$ by the formula $\Phi(a) := (a - \alpha(p(a)), p(a))$. It is a routine exercise to check that it is an isomorphism of semigroups, and thus (8.1) splits indeed. \square

The previous proof shows that (8.1) splits once we have a section of p . This may happen also for non-saturated affine semigroups, as we see by starting from a product $\Gamma^* \times \Gamma'$ between a lattice Γ^* and an arbitrary pointed affine semigroup Γ' . But such sections do not necessarily exist, as illustrated by Examples 2.11 and 2.12.

DEFINITION 8.2. The set of formal infinite sums:

$$(8.2) \quad \sum_{m' \in \Gamma'} a_{m'} \chi^{m'}, \quad a_{m'} \in K(\Gamma^*) \text{ for all } m',$$

with naturally defined addition and multiplication, is called **the ring of formal power series over Γ' with coefficients in $K(\Gamma^*)$** . We denote it by $K(\Gamma^*)[[\Gamma']]$.

REMARK 8.3. If L is a field and Γ is an affine semigroup, then the set $L[[\Gamma]]$ of formal power series with exponents in Γ is naturally a group by addition of coefficients, monomial-wise. But it becomes a ring by adding the intuitive multiplication law if and only if each element of Γ can be represented only *in a finite number of ways* as a sum of two elements of Γ , which is equivalent to the fact that Γ is pointed. This explains why we needed to work only with exponents in Γ' in the previous definition.

In the particular case when Γ is pointed, the sum (8.2) takes the simpler form:

$$\sum_{m \in \Gamma} a_m \chi^m$$

with $a_m \in K$. In this case, we write $K[[\Gamma]]$ instead of $K(\Gamma^*)[[\Gamma']]$ and we call this ring *the power series ring over Γ* .

EXAMPLE 8.4. If $\Gamma = \mathbb{Z}_{\geq 0}^n$, then the ring $K[[\Gamma]]$ is isomorphic to the ring $K[[x_1, \dots, x_n]]$ of formal power series in n variables with coefficients in K .

The semigroup Γ embeds naturally into the multiplicative semigroups of the rings $K[\Gamma]$ and $K[[\Gamma]]$. Moreover, a section $\alpha: \Gamma' \rightarrow \Gamma$ of p induces an embedding $\tilde{\alpha}: \Gamma \rightarrow K(\Gamma^*)[[\Gamma']]$:

$$\Gamma \ni m \mapsto \chi^{(m - \alpha(p(m)))} \cdot \chi^{\alpha(p(m))} \in K(\Gamma^*)[[\Gamma']],$$

(the monomial counterpart of the isomorphism Φ from the end of the proof of Lemma 8.1).

Notice that, if $\beta: \Gamma' \rightarrow \Gamma$ is another section of p , then $\tilde{\alpha}$ and $\tilde{\beta}$ differ by a unit, i.e., for any $m \in \Gamma \subset K(\Gamma^*)[[\Gamma']]$, there exists an element $u(m) \in \Gamma^*$ such that $\tilde{\beta}(m) = \chi^{u(m)}\tilde{\alpha}(m)$. In what follows, we consider also the localization $R = K[\Gamma]_{(\Gamma^+)}$ of the semigroup ring $K[\Gamma]$ at its ideal $(\Gamma^+) = (\{\chi^m \mid m \in \Gamma^+\})$. The semigroup $(\Gamma, +) \simeq (\chi^\Gamma, \cdot)$ is naturally also a subsemigroup of (R, \cdot) .

LEMMA 8.5. *Assume that the pointed affine semigroup Γ is such that p admits a section $\alpha: \Gamma' \rightarrow \Gamma$. Then α induces a unique isomorphism $\bar{\alpha}: K(\Gamma^*)[[\Gamma']] \rightarrow \widehat{R}_{\mathfrak{m}}$, where $\widehat{R}_{\mathfrak{m}}$ is the formal completion of the ring R at its maximal ideal $\mathfrak{m} = (\Gamma^+)$, such that the following diagram commutes:*

(8.3)

$$\begin{array}{ccc} & \Gamma & \\ \tilde{\alpha} \swarrow & & \searrow \\ K(\Gamma^*)[[\Gamma']] & \xrightarrow{\bar{\alpha}} & \widehat{R}_{\mathfrak{m}} \end{array}$$

PROOF. First, notice that a monomial $\chi^m \in \chi^\Gamma$ is contained in the ideal \mathfrak{m}^n if and only if $p(m) \in \Gamma'$ is contained in $n\Gamma'^+$. Next, diagram (8.3) shows that $\tilde{\alpha}$ is defined on the monomials. Notice also that $R = K[\Gamma]_{(\Gamma^+)} \simeq K(\Gamma^*)[[\Gamma']]_{(\Gamma'^+)}$. Recall that the ring $\widehat{R}_{\mathfrak{m}}$ is defined as the set of sequences $\{f_n\}_{n=1}^\infty$, $f_n \in R/\mathfrak{m}^n$, compatible with respect to the natural maps $R/\mathfrak{m}^n \rightarrow R/\mathfrak{m}^{n-1}$. If $f = \sum a_{m'}\chi^{m'} \in K(\Gamma^*)[[\Gamma']]$, then the sequence of its appropriate truncations defines a morphism of rings $K(\Gamma^*)[[\Gamma']] \rightarrow \widehat{R}_{\mathfrak{m}}$ which is obviously injective. To show surjectivity, write a representative for each f_n in the form b_n/c_n , where $b_n \in K[\Gamma]$, $c_n = d_n + q_n$, $d_n \in K[\Gamma^*]$, $d_n \neq 0$, $q_n \in (\Gamma^+)$. Since any $m \in \Gamma$ can be written as $u + \alpha(m')$ for some $u \in \Gamma^*$, $m' \in \Gamma'$, we may consider b_n and c_n as polynomials in $K(\Gamma^*)[\Gamma']$. Using the standard identity:

$$\frac{1}{1-q} \equiv 1 + q + \cdots + q^{n-1} \pmod{\mathfrak{m}^n}$$

for $q \in \mathfrak{m}$, we can rewrite:

$$\frac{b_n}{c_n} \equiv h_0 + h_1 + \cdots + h_{n-1} \pmod{\mathfrak{m}^n},$$

where:

$$h_k = \sum_{m' \in k\Gamma'^+ \setminus (k+1)\Gamma'^+} a_{m'}\chi^{m'} \in \mathfrak{m}^k \setminus \mathfrak{m}^{k+1}, \quad a_{m'} \in K(\Gamma^*) \text{ for all } m'.$$

Compatibility of the sequence $\{f_n\}$ implies that $\sum_{n=0}^\infty h_n$ is a well defined series from $K(\Gamma^*)[[\Gamma']]$. The last assertion of the lemma is obvious. \square

From Lemma 8.5, we deduce:

COROLLARY 8.6. *The ring $K(\Gamma^*)[[\Gamma']]$ is a local Noetherian domain, complete with respect to the $\hat{\mathfrak{m}}'$ -adic topology, where $\hat{\mathfrak{m}}'$ is its maximal ideal.*

In the remaining of this section we suppose that Γ is a pointed semigroup. Set $\sigma := \sigma(\Gamma)$. Consider a vector $w \in \sigma$. If:

$$f = \sum_{m \in \Gamma} a_m \chi^m \in K[[\Gamma]]$$

is a power series over Γ , $f \neq 0$, the w -order of f is:

$$(8.4) \quad w(f) = \min_{m: a_m \neq 0} \langle w, m \rangle,$$

and the w -initial form of f is:

$$\text{in}_w(f) = \sum_{m: \langle w, m \rangle = w(f)} a_m \chi^m \in K[\Gamma].$$

Note that if $w \in \circ$ (the interior of σ), then $\text{in}_w(f)$ is a polynomial. If I is an ideal of $K[[\Gamma]]$, the w -initial ideal $\text{in}_w(I)$ of I is the ideal generated by w -initial forms of all the elements of I . The same definitions can be given for the elements and the ideals of $K[\Gamma]$.

DEFINITION 8.7. The **extended Newton diagram** of a series $f \in K[[\Gamma]]$ is the set

$$\text{Newton}^+(f) = \text{Convex hull} \left(\bigcup_{m: a_m \neq 0} (m + \check{\sigma}) \right) \subseteq M(\Gamma) \otimes \mathbb{R}.$$

The **Newton diagram** $\text{Newton}(f)$ of f is the union of all compact faces of $\text{Newton}^+(f)$.

The extended Newton diagram of any series $f \in K[[\Gamma]]$ is a finite rational convex polyhedron, that is, it can be determined by finite number of linear inequalities of the form $\langle n, x \rangle \geq a$, $n \in N(\Gamma)$, $a \in \mathbb{Z}$. Moreover, $\text{Newton}^+(f)$ is contained in the cone $\check{\sigma}$ (the dual cone of σ) and this last cone is equal to the *recession cone* of $\text{Newton}^+(f)$ (which is defined as the maximal cone whose translation by any element of $\text{Newton}^+(f)$ is contained in $\text{Newton}^+(f)$).

More generally, if τ is a face of σ and if one takes $w \in \pi_\tau(\sigma) \subseteq \bar{\sigma}$ (see Section 4 and the formula (4.3)), then w defines a *preorder* (see the next section) on the monomials of the semigroup $\Gamma_\tau = \Gamma \cap \tau^\perp$. Thus, we can speak about w -initial forms and w -initial ideals for arbitrary weights w from $\bar{\sigma}$, but they should be applied to the τ -truncations of elements of $K[[\Gamma]]$ and understood as elements or ideals of the corresponding ring $K[[\Gamma_\tau]]$:

DEFINITION 8.8. Let Γ be a pointed affine semigroup. Let τ be any face of $\sigma(\Gamma)$ and $\Gamma_\tau = \Gamma \cap \tau^\perp$. If $f = \sum_{m \in \Gamma} a_m \chi^m \in K[[\Gamma]]$, the τ -**truncation** f_τ of f is defined by:

$$f_\tau = \sum_{m \in \Gamma \cap \tau^\perp} a_m \chi^m \in K[[\Gamma_\tau]].$$

The τ -**truncation** I_τ of an ideal $I \subset K[[\Gamma]]$ is defined as the ideal generated by the τ -truncations of its elements. If $w \in (N/N_\tau)_\mathbb{R}$, the w -**initial form** $\text{in}_w(f)$ of f is defined as $\text{in}_w(f_\tau)$. The w -**initial ideal** $\text{in}_w(I)$ of I is the ideal of the formal toric ring $K[[\Gamma_\tau]]$ which is generated by the w -**initial forms of its elements**.

It is easy to check that $K[[\Gamma_\tau]]$ is the quotient of $K[[\Gamma]]$ by the prime ideal $\Gamma \setminus \Gamma_\tau$, and the assignment $f \mapsto f_\tau$ gives the natural quotient homomorphism of rings $K[[\Gamma]] \rightarrow K[[\Gamma_\tau]]$.

9. Standard bases

In this section we explain the notion of *standard basis* of an ideal I in a formal power series ring $K[[\Gamma]]$ with respect to a *local monomial ordering*, which is a local analog of the notion of *Gröbner basis* of an ideal in a polynomial ring with respect to a monomial ordering. We prove the existence of a *universal standard basis*, that is, of a finite set of elements of I which are a standard basis with respect to *all* local monomial orderings.

As stated in Theorem 1.1 point 2), the global tropicalization of a subvariety of a torus can be obtained also by looking at the initial ideals of the defining ideal of the subvariety with respect to all weight vectors. Such weight vectors define preorders on the lattice of monomials of the torus, compatible with the addition. Usually they are studied by also bringing into the game total orderings compatible with the addition. Those total orderings allow to define the notion of *Gröbner basis* (see [CLOS 97], [CLOS 05], [Eis]). We refer to [BJSST] and [FJT] for their application to the study of tropicalization of subvarieties of tori.

Here, we develop an analogous theory of *standard basis* in formal power series rings $K[[\Gamma]]$, where Γ is an affine pointed semigroup. In the next two sections we use it to study the local tropicalizations of ideals in $K[[\Gamma]]$.

A *preorder* on a set is a binary relation which is both reflexive and transitive. A *partial order* is a preorder which is antisymmetric. A *total preorder* is a preorder such that any two elements of the set are comparable and a *total order* is a total preorder which is also a partial order. A *well ordered set* is a set endowed with a total order such that any nonempty subset has a minimum.

DEFINITION 9.1. A **local monomial ordering** on an affine pointed semigroup Γ is an order relation \preceq on the set Γ such that

- (i) 0 is the least element;
- (ii) \preceq is a total ordering;
- (iii) \preceq is compatible with addition on Γ , i. e., if $m \preceq m'$, then $m+n \preceq m'+n$ for any $n \in \Gamma$.

REMARK 9.2. If $m, n \in \Gamma$ and $m \preceq n$, we will also write $\chi^m \preceq \chi^n$. This explains the name *monomial ordering*: it is an order on the monomials of $K[[\Gamma]]$.

In the sequel, by a *monomial ordering* we shall always mean a local monomial ordering.

The following proposition is standard for $\Gamma \simeq \mathbb{N}^n$ (see [CLOS 97, Chapter 2.4, Cor. 6], where it is proved using the so-called *Dickson lemma* on finite generation by monomials of monomial ideals). We give here a proof which does not pass through an analog of Dickson's lemma.

LEMMA 9.3. *Under the axioms (ii) and (iii), condition (i) is equivalent to the fact that \preceq is a well-ordering of Γ .*

PROOF. Assume that \preceq is an ordering on Γ which satisfies the axioms (ii) and (iii). Suppose first that \preceq is a well-ordering of Γ . Arguing by contradiction, if (i) is not true, then there exists $m \in \Gamma$ such that $m \prec 0$.

Using axiom (ii), we get the following infinite chain of inequalities: $0 \succ m \succ 2m \succ 3m \succ \dots$. This implies that the set $\{0, m, 2m, 3m, \dots\}$ has no minimal element, which contradicts the hypothesis that we have a well-ordering.

Suppose then that axiom (i) is satisfied, in addition to (ii) and (iii). Choose a finite generating set $\{\gamma_i \mid i \in I\}$ of non-zero elements of Γ , which exists by the hypothesis that Γ is an affine semigroup. Assume by contradiction that Γ is not well-ordered. Then we get an infinite decreasing sequence $m_1 \succ m_2 \succ m_3 \succ \dots$ of elements of Γ . Choose also an expression $m_j = \sum_{i \in I} a_{ij} \gamma_i$ for each element of the sequence in terms of the chosen generating set. That is, $a_{ij} \in \mathbb{N}$ for all $i \in I, j \in \mathbb{N}^*$. Such expressions are in general not unique, but this does not matter here. By axiom (i), as the γ_i are non-vanishing, we see that $\gamma_i \succ 0, \forall i \in I$.

Consider now an arbitrary $j \geq 2$. As $m_1 \succ m_j$, by axiom (ii) there exists an index $i(j) \in I$ such that $a_{i(j),j} < a_{i(j),1}$. As the sequence $(m_j)_{j \geq 2}$ is infinite, we may extract an infinite subsequence in which $i(j)$ and $a_{i(j),j}$ are constant. Repeating this argument a finite number of times, we arrive at an infinite strictly decreasing sequence in which all the coefficients a_{ij} are constant when j varies, which is a contradiction. \square

DEFINITION 9.4. For a given monomial ordering \preceq on Γ , we define the **initial monomial** $\text{in}_{\preceq}(f)$ of any element f of $K[[\Gamma]]$ or $K[\Gamma]$ as the least monomial with non-zero coefficient in the expansion (8.2) of f and the **initial ideal** $\text{in}_{\preceq}(I)$ of an ideal I of $K[[\Gamma]]$ or $K[\Gamma]$ as the ideal generated by the initial monomials of all the elements of I .

Consider then any vector $w \in \sigma(\Gamma)$. We define a preorder relation \preceq_w on the elements of Γ (in fact of the whole $M(\Gamma)$) depending on w :

$$m \preceq_w m' \text{ if and only if } \langle w, m \rangle \leq \langle w, m' \rangle.$$

Note that $0 \preceq_w m$ for any $m \in \Gamma$, according to this definition. We say that a monomial ordering \preceq *refines* a preorder $\preceq_w, w \in \sigma$, if $m \preceq m'$ implies $m \preceq_w m'$.

Note that a monomial $\chi^m \in K[[\Gamma]]$ is *divisible* by χ^n if and only if $m = n + p$, where p is again an element of Γ . This implies that $n \preceq n + p$ for any monomial ordering on Γ .

The presence of a monomial ordering allows to extend the theory of divisibility from monomials to arbitrary series:

PROPOSITION 9.5. (Division algorithm) *Let \preceq be a fixed monomial ordering on Γ . If $f \in K[[\Gamma]]$ and $(f_1, \dots, f_p) \in K[[\Gamma]]^p$ is an ordered collection of series, then there are series $g_1, \dots, g_p, r \in K[[\Gamma]]$ such that:*

$$f = g_1 f_1 + \dots + g_p f_p + r,$$

where $\text{in}_{\preceq}(f) \leq \text{in}_{\preceq}(g_i f_i)$ for all i such that $g_i \neq 0, 1 \leq i \leq p$, and none of the monomials of r is divisible by any of the monomials $\text{in}_{\preceq}(f_1), \dots, \text{in}_{\preceq}(f_p)$.

PROOF. We simply apply the analog for series of the division algorithm for Gröbner basis (see [CLOS 97], [CLOS 05], [Eis]). Here this algorithm involves an infinite number of steps, which compute the coefficients of the unknown series g_1, \dots, g_p, r .

First, we find the smallest term $c \chi^m$, $c \in K$, $m \in \Gamma$, of f which is divisible by some $\text{in}_{\preceq}(f_i)$, $1 \leq i \leq p$. If i_0 is the first such i , we reduce f by defining:

$$R_1(f) := f - \frac{c \chi^m}{a \text{in}_{\preceq}(f_{i_0})} f_{i_0},$$

where $f_{i_0} = a \text{in}_{\preceq}(f_{i_0}) + \dots$. We repeat the same process with $R_1(f)$ instead of f , defining $R_2(f)$, and continue in the same way.

In the limit, we get a reduction $R_{\infty}(f)$ which has the property that no monomial of it is divisible by any $\text{in}_{\preceq}(f_i)$, $1 \leq i \leq p$. This is the remainder. Looking at the way we compute the sequence of reductions of f , we see that $f - R_{\infty}(f)$ is indeed of the form $g_1 f_1 + \dots + g_p f_p$, with $\text{in}_{\preceq}(f) \leq \text{in}_{\preceq}(g_i f_i)$ for all i such that $g_i \neq 0$, $1 \leq i \leq p$. \square

REMARK 9.6. This division result is usually presented for the ring of formal power series $K[[x_1, \dots, x_m]]$. An analogous (but more complicated) result for the ring of *convergent* power series $\mathbb{C}\{\{x_1, \dots, x_m\}\}$ was proved by Grauert (see [JP]), but we will not need it here.

Notice from the previous proof that the quotients g_1, \dots, g_p , as well as the remainder r , are uniquely determined by the process *if we carefully respect the order of the collection* f_1, \dots, f_p . But, in general, even the remainder changes if we change this order, as we show in the next example.

EXAMPLE 9.7. Take the ring $K[[x, y]]$ with the lexicographic ordering in which $x \prec y$, and the series $f = x, f_1 = x - y, f_2 = x - y^2$. Then, $\text{in}_{\preceq}(f_1) = \text{in}_{\preceq}(f_2) = x$, which shows that $g_1 = 1, g_2 = 0, r = y$. If we permute f_1 and f_2 , we get $r = y^2$.

This non-uniqueness of the remainder is eliminated if we take a *standard basis* instead of an arbitrary sequence (see Proposition 9.11).

DEFINITION 9.8. Let \preceq be a monomial ordering on Γ and I an ideal of $K[[\Gamma]]$. A finite sequence $\mathcal{B} \in I^p$ for some $p \in \mathbb{N}$ or, by abuse of language, the underlying set is called a **standard basis** for I with respect to the ordering \preceq if the initial monomials of the elements of \mathcal{B} generate the initial ideal $\text{in}_{\preceq}(I)$. A finite set $\mathcal{U} \subset I$ is called a **universal standard basis** for I if \mathcal{U} is a standard basis for I for any local monomial ordering \preceq on Γ .

REMARK 9.9. The terminology *standard basis* was introduced in [Hir, Chapter III.1] for a slightly different concept, not involving any ordering.

The existence of a standard basis for any ideal $I \subset K[[\Gamma]]$ and any monomial ordering \preceq on Γ follows from Noetherianness of $K[[\Gamma]]$ by a standard argument of the theory of Gröbner bases. The following three propositions are also standard.

PROPOSITION 9.10. *If \mathcal{B} is a standard basis for an ideal $I \subseteq K[[\Gamma]]$ with respect to some monomial ordering, then \mathcal{B} generates I .*

The next proposition shows that a standard basis induces a well-defined normal form for any element of $K[[\Gamma]]/I$. It corresponds to the remainder of the division by this basis.

PROPOSITION 9.11. *Let \preceq be a monomial ordering on Γ and I be an ideal of $K[[\Gamma]]$. Suppose that (f_1, \dots, f_p) is an associated standard basis. Consider the set $\Gamma_{\preceq}(I) \subset \Gamma$ of exponents of the monomials belonging to the monomial ideal $\text{in}_{\preceq}(I)$. Then, $\text{in}_{\preceq}(I)$ is generated as a semigroup ideal by the exponents of the initial monomials $\text{in}_{\preceq}(f_1), \dots, \text{in}_{\preceq}(f_p)$. Every element of $K[[\Gamma]]/I$ has a unique representative as a series whose monomials have exponents in the complement $\Gamma \setminus \Gamma_{\preceq}(I)$. This normal form is the remainder of the division algorithm by (f_1, \dots, f_p) .*

In particular, the remainder of the division of any element of I by a standard basis of I is necessarily 0. In fact, this characterizes standard basis:

PROPOSITION 9.12. *Let \preceq be a monomial ordering on Γ and I be an ideal of $K[[\Gamma]]$. Take $\mathcal{B} = (f_1, \dots, f_p) \in K[[\Gamma]]^p$. Then \mathcal{B} is a standard basis of I with respect to \preceq if and only if the remainder of the division of any element of I by \mathcal{B} is 0.*

The previous result allows to prove the following *stability* property of standard basis when we change the defining monomial ordering.

PROPOSITION 9.13. *Suppose that $\mathcal{B} = (f_1, \dots, f_p) \in K[[\Gamma]]^p$ is a standard basis of I with respect to the monomial ordering \preceq . If \preceq' is a second monomial ordering such that $\text{in}_{\preceq}(f_i) = \text{in}_{\preceq'}(f_i)$ for all $i \in \{1, \dots, p\}$, then \mathcal{B} is also a standard basis with respect to \preceq' .*

PROOF. Let $f \in I$. Divide f by \mathcal{B} with respect to \preceq' . Denote by $r \in I$ the remainder. By the previous proposition, it suffices to show that $r = 0$.

Suppose by contradiction that this is not the case. We know by Proposition 9.5 that no monomial of r is divisible by any monomial $\text{in}_{\preceq'}(f_i) = \text{in}_{\preceq}(f_i)$. But $r \in I$ and \mathcal{B} is a standard basis with respect to \preceq , which implies that:

$$\text{in}_{\preceq}(r) = \sum_{i=1}^k g_i \text{in}_{\preceq}(f_i)$$

for some $g_1, \dots, g_k \in K[[\Gamma]]$. This shows that the monomial $\text{in}_{\preceq}(r)$ is divisible by one of the monomials $\text{in}_{\preceq}(f_i)$, which is a contradiction. Thus $r = 0$. It follows that $\text{in}_{\preceq'}(f)$ is divisible by some $\text{in}_{\preceq}(f_i)$ and hence \mathcal{B} is a standard basis with respect to \preceq' . \square

The Newton polyhedron of f constrains deeply the possible initial terms of $f \in K[[\Gamma]]$ with respect to arbitrary monomial orderings of Γ .

LEMMA 9.14. *For any $f \in K[[\Gamma]]$, the exponent of the initial monomial $\text{in}_{\preceq}(f)$ is an element of the finite subset of Γ consisting of the vertices of $\text{Newton}(f)$.*

PROOF. Denote by $V(f)$ the set of vertices of $\text{Newton}^+(f)$ and by m_0 the exponent of $\text{in}_{\preceq}(f)$. There exists $n \in \text{Newton}(f) \cap M(\Gamma)_{\mathbb{Q}}$ with $m_0 - n \in \check{\sigma}(\Gamma)$. Indeed, take a half-line starting from m_0 and going to infinity inside $\text{Newton}^+(f)$ in a rational direction (that is, in direction of an element of Γ). Define then n as the intersection of the boundary of $\text{Newton}^+(f)$ with the opposite half-line. We have $n = m_0$ if and only if m_0 belongs to $\text{Newton}(f)$.

Consider now the canonical extension of \preceq to the whole rational vector space $M(\Gamma)_{\mathbb{Q}}$. We denote this extension by the same symbol \preceq . It can be constructed in the same way as we construct the extension to \mathbb{Q} of the usual order on \mathbb{N} : extend it first to $M(\Gamma)$ by setting $m_1 - m_2 \succ 0 \Leftrightarrow m_1 \succ m_2$ for any $m_1, m_2 \in \Gamma$, then to $M(\Gamma)_{\mathbb{Q}}$ by setting $\lambda \cdot m \succ 0$ for any $\lambda \in \mathbb{Q}_+^*$ and any $m \in M(\Gamma)$ such that $m \succ 0$. It is a routine exercise to verify that we get like this a well-defined total order on $M(\Gamma)_{\mathbb{Q}}$.

Let us come back to the exponents $m_0 \in \Gamma$ and to $n \in \check{\sigma}(\Gamma) \cap \Gamma_{\mathbb{Q}}$. As, by construction, $m_0 - n$ is positively proportional to an element of Γ , we get the inequality $m_0 \succeq n$.

Choose now an arbitrary face P of the Newton diagram $\text{Newton}(f)$ containing n . It is a compact convex polyhedron in $M(\Gamma)_{\mathbb{R}}$, with vertices in Γ and with dimension at most $\text{rk}(M(\Gamma)) - 1$. If $(v_j)_{j \in J}$ is the set of its vertices, we have therefore a convex expression of n in terms of those vertices:

$$n = \sum_{j \in J} p_j \cdot v_j, \text{ with } \sum_{j \in J} p_j = 1 \text{ and } p_j \in [0, 1] \text{ for all } j \in J.$$

Let v_0 be the minimal vertex of P with respect to \preceq . Then, as all the coefficients p_j are non-negative, we deduce from the compatibility of \preceq with the \mathbb{Q} -vector space structure of $M(\Gamma)$ that $n = \sum_{j \in J} p_j \cdot v_j \succeq \sum_{j \in J} p_j \cdot v_0 = v_0$. Combining this inequality with the inequality $m_0 \succeq n$ obtained before, we get $m_0 \succeq v_0$. As m_0 is by definition the exponent of $\text{in}_{\preceq}(f)$, we deduce that $m_0 = v_0$, which proves the lemma. \square

COROLLARY 9.15. *Let I be an ideal of $K[[\Gamma]]$, \preceq a monomial ordering on Γ , and $\mathcal{B} = \{f_1, \dots, f_k\}$ a standard basis of I with respect to \preceq . Let \preceq' be a second monomial ordering which coincides with \preceq when restricted to the finite set $\{m \in \Gamma \mid \exists i = 1, \dots, k: m \in \text{Newton}(f_i)\}$. Then \mathcal{B} is also a standard basis with respect to \preceq' .*

PROOF. By Lemma 9.14, we have $\text{in}_{\preceq}(f_i) = \text{in}_{\preceq'}(f_i)$ for all $i \in \{1, \dots, k\}$, which implies the desired assertion by Proposition 9.13. \square

As a consequence, a standard basis for a monomial order remains standard for conveniently defined *neighboring* orders. Following Sikora [Sik], Boldini [Bol09] and [Bol10], we see now that there is indeed a notion of topology on the space of monomial orders such that standard bases are locally constant.

Let S be any set. Denote by:

$$TO(S)$$

the set of all total orderings of S . One has a natural topology on it. Intuitively, given two elements $a, b \in S$ such that $a \prec b$ for some ordering $\preceq \in TO(S)$, then this strict inequality should also hold in a neighborhood of \preceq . Therefore, one is forced to declare the subsets:

$$U_{(a,b)} := \{\preceq \in TO(S) \mid a \preceq b\}$$

open, for all $a, b \in S$. Therefore, we endow $TO(S)$ with the topology generated by them.

In the case when S is a semigroup and we only take the orderings that are compatible with the semigroup law, this topology was defined by Sikora

[Sik]. The extension to arbitrary sets was done by Boldini [Bol09]. Sikora proved that under the additional hypothesis that S is countable the associated topology is compact. Boldini proved the analogous fact for an arbitrary countable set:

PROPOSITION 9.16 ([Bol09, Theorem 1.4]). *If the set S is countable, then the space $TO(S)$ is compact.*

Given an element $a \in S$, let $SO_a(S)$ be the subspace of $TO(S)$ consisting of all total orderings for which the element a is minimal, i.e., $a \leq b \forall b \in S$.

PROPOSITION 9.17 ([Bol09, Theorem 1.5]). *The subspace $SO_a(S)$ is closed in $TO(S)$ for each $a \in S$. Hence, if S is countable, $SO_a(S)$ is compact.*

Now we let $S = \Gamma$ be an affine pointed semigroup. We denote by:

$$MO(\Gamma)$$

the set of all monomial orderings on Γ .

PROPOSITION 9.18 ([Bol09, Theorem 2.4]). *$MO(\Gamma)$ is a closed compact subset of $SO_0(\Gamma)$.*

LEMMA 9.19 (cf. [Bol09, Lemma 2.10]). *Let I be an ideal of $K[[\Gamma]]$, and \mathcal{B} a finite subset of I . Then, the set of all monomial orderings \preceq such that \mathcal{B} is a standard basis with respect to \preceq is open in $MO(\Gamma)$.*

PROOF. The proof is essentially the same as in [Bol09]. In view of Lemma 9.15, we must only replace the support of \mathcal{B} with the set of monomials of the series from \mathcal{B} lying on the union of Newton diagrams of elements of \mathcal{B} . \square

It is not obvious from the definition that universal standard basis indeed exist. Nevertheless, it is an immediate consequence of the compactness of the space of monomial orderings:

THEOREM 9.20. *Any ideal of the ring $K[[\Gamma]]$ has a universal standard basis.*

PROOF. Our argument is similar to that of [Bol09, Theorem 2.14]. The family $U_{\mathcal{B}}$, where \mathcal{B} runs over all finite subsets of I , forms an open covering of the space $MO(\Gamma)$. Since $MO(\Gamma)$ is compact, we can choose a finite subcovering $U_{\mathcal{B}_1}, \dots, U_{\mathcal{B}_k}$. We conclude that the union $\cup_{i=1}^k \mathcal{B}_i$ is a universal standard basis of I . \square

The following proposition will be used in the proof of Theorem 10.3:

PROPOSITION 9.21. *If $\mathcal{U} = \{f_1, \dots, f_p\}$ is a universal standard basis for an ideal $I \subset K[[\Gamma]]$ and $w \in \sigma(\Gamma)$, then $\text{in}_w(\mathcal{U}) = \{\text{in}_w(f_1), \dots, \text{in}_w(f_p)\}$ is a universal standard basis for the initial ideal $\text{in}_w(I)$.*

PROOF. We have to show that $\text{in}_w(\mathcal{U})$ is a standard basis of $\text{in}_w(I)$ for any monomial ordering. Let $g \in \text{in}_w(I)$, and \preceq_{α} be a monomial ordering on Γ . Consider the ordering $\preceq_{w, \alpha}$ which is defined by comparing the monomials first by \preceq_w , and then by \preceq_{α} . Clearly the initial terms of $\text{in}_w(f_1), \dots, \text{in}_w(f_p)$, g with respect to \preceq_{α} and with respect to $\preceq_{w, \alpha}$ coincide. On the other hand,

since \mathcal{U} is universal, $\text{in}_{w,\alpha}(g)$ is divisible by at least one of $\text{in}_{w,\alpha}(f_i)$ (we use here also the fact that for each $g \in \text{in}_w(I)$ there exists $f \in I$ such that $\text{in}_w(g) = \text{in}_w(f)$, see Lemma 9.22 below). This concludes our proof. \square

LEMMA 9.22. *Suppose that $w \in \sigma(\Gamma)$. Then, for all $h \in \text{in}_w I$, there exists $f \in I$ with $\text{in}_w h = \text{in}_w f$.*

PROOF. Since $h \in \text{in}_w I$, there exist $h_1, \dots, h_n \in K[[\Gamma]]$ and $f_1, \dots, f_n \in I$ such that:

$$(9.1) \quad h = h_1 \text{in}_w(f_1) + \dots + h_n \text{in}_w(f_n).$$

Notice that w , considered as a morphism of semigroups from Γ to $\mathbb{R}_{\geq 0}$, has a countable image with infinity as the single accumulation point. Thus we may write:

$$\text{Im}(w) = \{\mu_0 = 0, \mu_1, \mu_2, \dots\},$$

where $\mu_0 < \mu_1 < \mu_2 < \dots$. Now every series $g \in K[[\Gamma]]$ can be decomposed into its weighted homogeneous components g_{μ_i} , $w(g_{\mu_i}) = \mu_i$:

$$g = \sum_{i=0}^{\infty} g_{\mu_i}.$$

Each g_{μ_i} is a w -weighted homogeneous series. Applying such a decomposition to (9.1) and comparing forms of w -order $w(h)$, we get:

$$\text{in}_w(h) = \sum_{j=1}^n h_{j,w(h)-w(f_j)} \text{in}_w(f_j),$$

where $h_{j,w(h)-w(f_j)}$ is the w -homogeneous component of h_j of order $w(h) - w(f_j)$ (it is 0 by definition if $w(h) - w(f_j) < 0$).

Now consider the following element of I :

$$f = \sum_{j=1}^n h_{j,w(h)-w(f_j)} f_j.$$

Each f_j , $1 \leq j \leq n$, has the form:

$$f_j = \text{in}_w(f_j) + (\text{terms of order } > w(f_j)).$$

It follows that $\text{in}_w(f) = \text{in}_w(h)$. \square

10. Tropical bases

As in the case of subvarieties of tori [BJSST, Theorem 11], in this section we prove the existence of tropical bases of ideals of $K[[\Gamma]]$. These bases are particular systems of generators of I that allow us to compute $\text{Trop}_{>0}(I)$ as the intersection of local tropicalizations of hypersurfaces defined by these generators.

DEFINITION 10.1. Let Γ be a pointed affine semigroup. A **tropical basis** of an ideal I of the ring $K[[\Gamma]]$ is a universal standard basis $\{f_1, \dots, f_p\}$ of I such that for any $w \in \sigma(\Gamma)$, the ideal $\text{in}_w(I)$ contains a monomial if and only if one of the initial terms $\{\text{in}_w(f_1), \dots, \text{in}_w(f_p)\}$ is a monomial.

It is not always true that a universal standard basis is tropical:

EXAMPLE 10.2. We consider the same polynomials as the ones chosen in [BJSST, Example 10], but this time seen as generators of an ideal of the ring $\mathbb{C}[[x, y, z]]$ of formal power series in three variables. Namely, we take:

$$I = (x + y + z, xy(x + y), xz(x + z), yz(y + z)).$$

Any two of the last three polynomials are redundant as generators, but they are needed in order to get a universal standard basis. We show that the four generators of I form a universal standard basis of I .

Consider an arbitrary monomial ordering \preceq . We have to show that the initial ideal $\text{in}_{\preceq}(I)$ is generated by the initial terms of these four polynomials. Since the set is symmetric in x, y, z , it is enough to study the case when $x \prec y \prec z$. Therefore, we have to show that $\text{in}_{\preceq}(I)$ is generated by the monomials x and y^2z ($y \prec z$ implies $y^2z \prec yz^2$).

Which monomials are not in the ideal generated by x and y^2z ? Only those of the form y^k, z^k , and yz^k . Let us show that none of these belongs to $\text{in}_{\preceq}(I)$.

Consider first the case of yz^k . If $yz^k \in \text{in}_{\preceq}(I)$, then there exist two series $f, g \in \mathbb{C}[[x, y, z]]$ such that yz^k is the \preceq -initial term of:

$$(10.1) \quad f \cdot (x + y + z) + g \cdot (y^2z + yz^2),$$

since $x + y + z$ and $y^2z + yz^2$ generate I . First, let us substitute $x = 0$ (take the quotient $k[[x, y, z]]/(x)$ and consider the induced monomial order on it). We get:

$$\begin{aligned} yz^k &= \text{in}_{\preceq}(f_0 \cdot (y + z) + g_0 \cdot (y^2z + yz^2)) = \\ &= \text{in}_{\preceq}((y + z)(f_0 + yzg_0)) = y \text{in}_{\preceq}(f_0 + yzg_0). \end{aligned}$$

It follows that the initial term of f_0 is z^k , and hence z^k has a non-zero coefficient in f too.

Therefore, when we distribute the product $f \cdot (x + y + z)$, we get the monomial xz^k as a term in this expansion. But $xz^k \prec yz^k$. Therefore, it must cancel in (10.1). As this monomial does not appear in $g \cdot (y^2z + yz^2)$, we see that xz^k cancels only if f contains also the monomial x^2z^{k-1} . Again, then the product $f \cdot (x + y + z)$ contains x^2z^{k-1} which is less than xz^k and yz^k . We conclude that f contains also x^3z^{k-2} , and so on. But then we come to a contradiction, because the series f does not have any negative powers of z .

The argument for y^k and z^k is similar and even easier, because we do not need to pass to the quotient $k[[x, y, z]]/(x)$.

The fact that the four polynomials are not a tropical basis is proved now exactly as in [BJSST, Example 10]. Namely, consider the weight $w = (1, 1, 1)$. The four polynomials are equal to their initial terms with respect to w (they are homogeneous), therefore these initial terms are not monomials. But $xyz \in I$, therefore $\text{in}_w(I)$ contains the monomial xyz . This shows that the four polynomials do not form a tropical basis of I .

Therefore, we are led to ask whether tropical bases for ideals of rings of the form $K[[\Gamma]]$ exist necessarily. This is indeed the case:

THEOREM 10.3. *Any ideal of the ring $K[[\Gamma]]$ has a tropical basis.*

PROOF. Starting from any universal standard basis $\mathcal{U} = \{f_1, \dots, f_k\}$ of a given ideal I , we shall construct a tropical basis of I by adding new series to \mathcal{U} .

Note that the cone $\sigma = \sigma(\Gamma)$ is naturally stratified by the relative interiors of its faces:

$$\sigma = \bigsqcup_{\tau \leq \sigma} \overset{\circ}{\tau}.$$

Furthermore, if $f \in K[[\Gamma]]$, each $w \in \sigma$ can be considered as a function on the extended Newton diagram $\text{Newton}^+(f)$ of f . This function takes its minimal value on some face of $\text{Newton}^+(f)$. We say that this face is *cut by the function* w . Now, we define an equivalence relation on the set of vectors of the cone σ : $w \sim w'$ if and only if w and w' are contained in the same stratum $\overset{\circ}{\tau}$ and for all i , $1 \leq i \leq k$, w and w' cut the same face of $\text{Newton}^+(f_i)$. The reader can easily check that there are only finite number of equivalence classes of \sim , that they give a new stratification of σ refining the one described above, and the closure of each equivalence class is a rational polyhedral cone. The set of these cones is a fan that we denote by $\Sigma_{\mathcal{U}}$. Moreover, it follows from Proposition 9.21 that if $w \sim w'$, then $\text{in}_w(I) = \text{in}_{w'}(I)$.

Thus, $\Sigma_{\mathcal{U}}$ is a refinement of the *local Gröbner fan of the ideal* I . This notion was introduced by Bahloul and Takayama in [BT 04], [BT 07] for ideals of formal power series rings $K[[X_1, \dots, X_n]]$ as a local analog of the notion of *Gröbner fan of an ideal* of a polynomial ring introduced by Mora and Robbiano [MR]. It may be immediately extended in our context.

Let ρ be a cone of $\Sigma_{\mathcal{U}}$ such that for some (and thus for any) $w \in \overset{\circ}{\rho}$ the initial ideal $\text{in}_w(I)$ contains a monomial. If $\text{in}_w(f_i)$ is a monomial for some $f_i \in \mathcal{U}$, we do not add any series to \mathcal{U} . Assume then that none of $\text{in}_w(f_i)$ is a monomial and let $\chi^m \in \text{in}_w(I)$ be a monomial. Choose an irrational point $w' \in \sigma$, so that the preorder determined by the vector w' is actually a monomial ordering. Let $\preceq_{w,w'}$ be a monomial ordering defined by comparing the monomials first by \preceq_w and then by $\preceq_{w'}$. Now, divide the monomial χ^m by \mathcal{B} with respect to $\preceq_{w,w'}$ (see Proposition 9.5). We get an expression:

$$\chi^m = \sum_i g_i f_i + r.$$

Notice that the initial monomials of f_1, \dots, f_k with respect to $\preceq_{w,w'}$ are independent of w (whenever $w \in \overset{\circ}{\rho}$). It follows that the remainder r is also independent of w (the reason is that the monomials not contained in the initial ideal $\text{in}_{\preceq_{w,w'}}(I)$ form a “basis” of the quotient ring $K[[\Gamma]]/I$, see the proof of Lemma 11.10). Also, since χ^m can be represented as a combination of w -initial forms of f_1, \dots, f_k , the value $w(r)$ of r is strictly greater than $w(\chi^m) = \langle w, m \rangle$, and this also holds for all $w \in \overset{\circ}{\rho}$. The element $f_{\rho} = \chi^m - r$ lives in I , and by construction the w -initial form of f_{ρ} is the monomial χ^m , for any $w \in \overset{\circ}{\rho}$. Adding to \mathcal{U} all the series of the form f_{ρ} , $\rho \in \Sigma_{\mathcal{U}}$, as described above, we get a tropical basis for I . \square

Finally, we generalize the notion of tropical basis so that it allows also to study the initial ideals corresponding to arbitrary, not necessarily finite, vectors $w \in \overline{\sigma}(\Gamma)$. If τ is a face of $\sigma(\Gamma)$, we consider the τ -truncation I_{τ} of

the ideal I (see Definition 8.8). It is easy to see that any element of I_τ is a truncation of some element of I . For each τ , let us choose a finite set \mathcal{B}_τ of elements of I such that the set of truncations of \mathcal{B}_τ is a tropical basis for I_τ . Setting:

$$\mathcal{B} = \bigcup_{\tau \leq \sigma} \mathcal{B}_\tau,$$

we get a finite subset of I such that its truncation in every $K[[\Gamma_\tau]]$ is a tropical basis of I_τ . We turn this property into a definition:

DEFINITION 10.4. A finite subset of an ideal I of $K[[\Gamma]]$, such that its truncation in every ring $K[[\Gamma_\tau]]$ for varying faces τ of $\sigma(\Gamma)$ is a tropical basis of I_τ is called an **extended tropical basis** of I .

An extended tropical basis \mathcal{B} may be characterized also by the property that for any face τ of σ and any $w \in \bar{\sigma} \cap (N/N_\tau)_\mathbb{R}$, the initial ideal $\text{in}_w(I)$ (considered as an ideal of the ring $K[[\Gamma_\tau]]$) contains a monomial if and only if one of the initial forms $\text{in}_w(f)$, $f \in \mathcal{B}$, is a monomial.

11. The local finiteness theorem

Our main goal here is to describe the piecewise-linear structure of the local tropicalization. We were not able to prove this fact in full generality. We could do this only for quotient rings of the ring of formal power series $K[[\Gamma]]$ over a pointed affine semigroup Γ (see Theorem 11.9) and for another related class of morphisms (see Theorem 11.14).

In this section we keep the assumption that Γ is an affine pointed semigroup. As usual, $\sigma = \sigma(\Gamma)$. Recall from Remark 6.9 that we denote:

$$\text{Trop}_{>0}(I) := \text{Trop}_{>0}(\gamma) \text{ and } \text{Trop}_{\geq 0}(I) := \text{Trop}_{\geq 0}(\gamma)$$

if I is an ideal of the ring $K[[\Gamma]]$ and $\gamma: \Gamma \rightarrow K[[\Gamma]]/I$ is the natural semigroup morphism. We start proving that the definitions through extensions of valuations and initial ideals lead to the same concept of local tropicalization for the canonical morphism of semigroups $(\Gamma, +) \rightarrow (K[[\Gamma]], \cdot)$. The following result plays an essential role in the proof of Theorem 11.2.

THEOREM 11.1 ([Berg1, Corollary 1]). *Let R be a commutative ring with unit and v a valuation on R . Let I be an ideal of R and S a multiplicative subsemigroup of (R, \cdot) such that there is no $g \in S$, $f \in I$ satisfying $v(g) = v(f) < v(f - g)$. Then, there exists a valuation $v' \geq v$ on R such that $v'|_I = +\infty$, $v'|_S = v|_S$.*

We now apply Theorem 11.1 to local tropicalizations.

THEOREM 11.2. *Suppose that Γ is an affine pointed semigroup. Let I be an ideal of the ring $K[[\Gamma]]$ and denote by $\gamma: \Gamma \rightarrow K[[\Gamma]]/I$ the natural semigroup morphism. Then:*

- (i) $\text{Trop}_{>0}(I) = \text{Trop}_{\geq 0}(I) \cap \bar{\sigma}^\circ$.
- (ii) *The following two subsets of the linear variety $L(\sigma, N)$ coincide:*
 - (1) *the local nonnegative tropicalization $\text{Trop}_{\geq 0}(I)$;*
 - (2) *the set T of those $w \in \bar{\sigma} \subset L(\sigma, N)$ such that the initial ideal $\text{in}_w(I)$ is monomial free.*

PROOF. Part (i) follows from Proposition 6.8 (ii), so let us prove the second part. First we show that $\text{Trop}_{\geq 0}(\gamma) \subseteq T$. Any valuation v of $K[[\Gamma]]/I$ lifts to a valuation \bar{v} of $K[[\Gamma]]$ such that $\bar{v}|_I = +\infty$. Consider the vector $w \in L(\sigma, N)$ determined by the valuation \bar{v} . If v is nonnegative on $K[[\Gamma]]/I$, then the vector w is contained in $\bar{\sigma}$. Consider an arbitrary $f \in I$. Since $\bar{v}(f) = +\infty$, w takes its minimal value on at least two monomials in f . Therefore, $\text{in}_w(f)$ is not a monomial. As f was chosen arbitrarily inside I , we see that indeed $\text{in}_w(I)$ is monomial free.

Now let us show that $T \subseteq \text{Trop}_{\geq 0}(\gamma)$. Choose any $w \in \bar{\sigma}$ such that $\text{in}_w(I)$ is monomial free. The extended weight vector w defines a monomial valuation on $K[[\Gamma]]$ or on $K[[\Gamma_\tau]]$ if w belongs to a stratum at infinity of $L(\sigma, N)$: $f \mapsto w(f)$ (see (8.4)). In the latter case we also want to consider w as a valuation on the whole of $K[[\Gamma]]$. For this, if $w \in (N/N_\tau)_\mathbb{R}$ and $f = \sum_{m \in \Gamma} a_m \chi^m$, first take the τ -truncation f_τ (see Definition 8.8), and then apply the valuation w . Note that for any $f \in I$ and $m \in \Gamma$, it is impossible to have simultaneously $w(\chi^m) = w(f) < w(f - \chi^m)$ because f has at least two monomials χ^m and χ^n of minimal value (as $\text{in}_w(I)$ is supposed monomial free). Thus, by Theorem 11.1, there exists a valuation on $K[[\Gamma]]/I$ giving exactly the point w under the tropicalization map. \square

REMARK 11.3. Consider any point $w \in \text{Trop}_{> 0}(I) \cap N$. Then, by Theorem 10.3 and Theorem 11.2 (ii), in a neighborhood of w the positive tropicalization of an ideal I coincides with that of the initial ideal $\text{in}_w(I)$.

In the sequel we want to make a clear distinction between a *fan* and a set which is the *support of a fan*, but without fixed fan structure. That is why we introduce the following definition:

DEFINITION 11.4. A **PL cone** in a vector space L is a subset $\Sigma \subseteq L$ that can be represented as a finite union of convex polyhedral cones. It is called **rational**, if it can be represented as a finite union of rational convex polyhedral cones, that is, if it is the support of a fan. A subset Σ of a linear variety $L(\sigma, N)$ is called a **(rational) PL conical subspace** if for each stratum $(N/N_\tau)_\mathbb{R}$ the intersection $\Sigma \cap (N/N_\tau)_\mathbb{R}$ is empty or a (rational) PL cone.

The following are examples of PL cones and PL conical subspaces:

DEFINITION 11.5. The **Newton cone** of a series $f \in K[[\Gamma]]$, denoted $\text{Newton}^\perp(f)$, is the set of vectors $w \in \sigma$ such that, seen as a function on the Newton diagram $\text{Newton}(f)$, w attains its minimum on a face of positive dimension. If $f = 0$, we set $\text{Newton}^\perp(f) = \sigma$ by definition. The **extended Newton cone** of f is the disjoint union $\text{Newton}^\perp(f) := \bigsqcup_{\tau \leq \sigma} \text{Newton}^\perp(f_\tau)$, where f_τ is the τ -truncation of f .

If $f \neq 0$ and $f \neq u \cdot \chi^m$, where $m \in \Gamma$ and u is a unit in $K[[\Gamma]]$, then the Newton cone is indeed a rational PL conical subspace of pure dimension $n - 1$, where n is the rank of the lattice $N = N(\Gamma)$. Notice that our Newton cone is different from what is usually called the *normal fan* of f . The Newton cone is the support of the $(n - 1)$ -skeleton of the standard normal fan.

EXAMPLE 11.6. Consider $\Gamma = \mathbb{N}^2 \subset \mathbb{Z}^2 = M$ and the reducible polynomial $f = x(x + y)(x + y^2) = x^3 + x^2y + x^2y^2 + xy^3 \in K[[\Gamma]] = K[[x, y]]$.

Here we set $x = \chi^{(1,0)}$ and $y = \chi^{(0,1)}$, where $e_1 = (1, 0), e_2 = (0, 1)$ form the canonical basis of \mathbb{Z}^2 . In Figure 5 are represented its associated extended Newton diagram, Newton cone and extended Newton cone. The black discs in the drawing of the extended Newton diagram represent the exponents of the monomials of f . The Newton diagram $\text{Newton}(f)$ has two edges, denoted AB and BC in the figure, where $A = (3, 0), B = (2, 1), C = (1, 3)$. The Newton cone $\text{Newton}^\perp(f)$ lives in the dual plane $\mathbb{R}^2 = N_{\mathbb{R}}$, endowed with the dual basis (v_1, v_2) of (e_1, e_2) . It is contained in the cone $\sigma = \mathbb{R}_+ v_1 + \mathbb{R}_+ v_2$, whose edges are $\tau_i = \mathbb{R}_+ v_i$ for $i = 1, 2$. It is the union of two closed half-lines, H_{AB} normal to AB and H_{BC} normal to BC . The extended Newton cone $\text{Newton}^\perp(f)$ lives in the affine linear variety $L(\sigma, N)$. In addition to H_{AB} and H_{BC} it contains at infinity the half-line H_1 , projection of σ to $L_1 = (N/N_{\tau_1})_{\mathbb{R}}$, and the point $L_{12} = (N/N_\sigma)_{\mathbb{R}}$. Note that the closures of the three half-lines H_{AB}, H_{BC}, H_1 contain the point L_{12} at infinity.

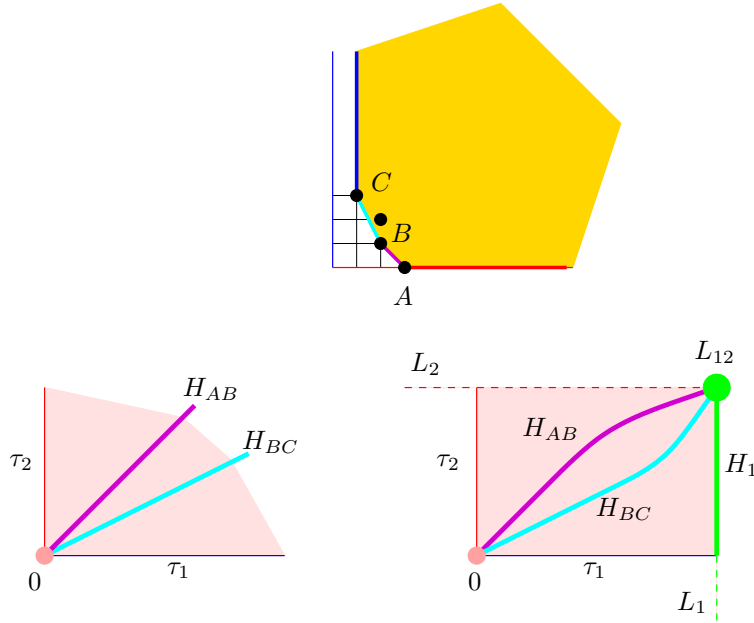


FIGURE 5. An extended Newton diagram and its associated Newton cone and extended Newton cone in dimension 2

The extended Newton cone of f can be connected to the closure of the Newton cone $\text{Newton}^\perp(f)$:

LEMMA 11.7. *Let $f \in K[[\Gamma]]$, and let \mathfrak{m}_τ denote the ideal of $K[[\Gamma]]$ generated by all the monomials with exponents outside τ^\perp , where τ is a face of σ . Then:*

$$\overline{\text{Newton}^\perp(f)} \cap (N/N_\tau)_{\mathbb{R}} \subseteq \text{Newton}^\perp(f_\tau)$$

with equality if f is not contained in \mathfrak{m}_τ . Here, $\overline{(\cdot)}$ denotes the closure in $\overline{\sigma}$ and each $\text{Newton}^\perp(f_\tau)$ is naturally embedded in the stratum $(N/N_\tau)_{\mathbb{R}}$. The

only cases when f is contained in \mathfrak{m}_τ but still there is an equality in the formula above is when $f = 0$, $\tau = \{0\}$ or $\tau = \sigma$. In particular, we have:

$$\overline{\text{Newton}^\perp(f)} \subseteq \bigsqcup_{\tau \leq \sigma} \text{Newton}^\perp(f_\tau)$$

with equality if and only if f is not contained in any of \mathfrak{m}_τ , $\tau \leq \sigma$.

PROOF. If $f_\tau = 0$, the inclusion is trivial since $\text{Newton}^\perp(f_\tau)$ is the projection of the cone σ to $(N/N_\tau)_\mathbb{R}$. The equality in the cases $f = 0$, $\tau = \{0\}$ or $\tau = \sigma$ can be checked directly. The reason why these are the only exceptions is that if $f \neq 0$, then $\text{Newton}^\perp(f)$ is a *proper* PL cone in σ , and its limit points in $(N/N_\tau)_\mathbb{R}$ cannot be the whole projection of σ unless $\tau = \{0\}$ or $\tau = \sigma$.

Now assume that $f \notin \mathfrak{m}_\tau$, so that $f_\tau \neq 0$. Let v^τ be an element of $\overline{\sigma} \cap (N/N_\tau)_\mathbb{R}$ and $(v_n)_{n \in \mathbb{N}}$ a sequence of elements of $\text{Newton}^\perp(f)$ converging to v^τ . This implies that the corresponding sequence $(v_n^\tau)_{n \in \mathbb{N}}$ of projections of $(v_n)_{n \in \mathbb{N}}$ to $(N/N_\tau)_\mathbb{R}$ converges to v^τ in the usual sense. Let μ be the minimal value of v^τ considered as a function on the Newton diagram $\text{Newton}(f_\tau)$. Note that if ρ is the face of $\text{Newton}(f_\tau)$ where v^τ attains its minimum, then starting from some number $n_0 \in \mathbb{N}$, for $i \geq n_0$ each function v_i^τ reaches its minimal value on some face of ρ .

Let U be the open subset of $N_\mathbb{R}$ formed by all functions v that take values strictly greater than μ on all monomials of f lying on $\text{Newton}(f)$, except possibly those lying on τ^\perp . Then \overline{U}_τ (see (4.3)) is an open neighborhood of v^τ and all elements v_i are contained in \overline{U}_τ for $i \geq n_1$, for sufficiently large number n_1 . It follows that all v_i attain their minimal values on the faces of $\rho \subseteq \text{Newton}(f_\tau) = \text{Newton}(f) \cap \tau^\perp$ for $i \geq \max\{n_0, n_1\}$. Thus, the face ρ has positive dimension and $v_\tau \in \text{Newton}^\perp(f_\tau)$. This shows that:

$$\overline{\text{Newton}^\perp(f)} \cap (N/N_\tau)_\mathbb{R} \subseteq \text{Newton}^\perp(f_\tau).$$

If $\text{Newton}^\perp(f_\tau) = \emptyset$, then the equality holds. Assume that $\text{Newton}^\perp(f_\tau)$ is nonempty. It follows that there are at least 2 vertices of $\text{Newton}(f)$ lying on the linear subspace τ^\perp . Let $\overline{v} \in \text{Newton}^\perp(f_\tau)$ and ρ a face of $\text{Newton}(f_\tau) \subseteq \text{Newton}(f)$ where \overline{v} attains its minimum. Let $v \in \sigma$ be any element that projects to \overline{v} under the canonical projection $N_\mathbb{R} \xrightarrow{\pi} (N/N_\tau)_\mathbb{R}$. Now let $u \in \tau \setminus \{0\}$ be any vector and consider the sequence $v_n = v + nu$, $n \geq 0$. For n big enough v_n attains its minimum on some face of $\text{Newton}(f) \cap \tau^\perp$. This face must be ρ , $v_n \in \text{Newton}^\perp(f)$ for $n \gg 0$, and $v_n \rightarrow \overline{v}$ in $L(N, \sigma)$. \square

The following proposition describes the local tropicalization in the hypersurface case. It should be compared with [EKL, Theorem 2.1.1] and [BG, Section 3], where a description of the global tropicalization of a hypersurface is given. The languages used in these papers are different from ours, still we can interpret them in the language of general tropicalization developed by us in Section 6. To get the setting of [BG], and [EKL], take Γ to be a free finitely generated abelian group G , R the coordinate ring of a subvariety of the torus $\text{Spec } K[G]$, and $\gamma: G \rightarrow R$ the natural morphism. The field K is supposed to be endowed with an arbitrary (not necessarily trivial) real valuation. This leads to a global tropicalization that is a

PL (but not necessarily conical) subspace. Notice that in the case of local tropicalization we must restrict to trivially valued fields. Indeed, if v is a nonnegative valuation on a ring R , then v must be trivial on any subfield $K \subseteq R$.

PROPOSITION 11.8 (Local tropicalization in the hypersurface case). *Let $f \in K[[\Gamma]]$ be a non-invertible series. Then, the nonnegative tropicalization $\text{Trop}_{\geq 0}(f)$ of the natural semigroup morphism $\Gamma \rightarrow K[[\Gamma]]/(f)$ coincides with the extended Newton cone of f . The closure of the positive tropicalization $\text{Trop}_{> 0}(f) \subseteq \bar{\sigma}^\circ$ in $\bar{\sigma}$ is $\text{Trop}_{\geq 0}(f)$.*

PROOF. Apply Theorem 11.2 and Lemma 11.7 to the principal ideal $I = (f)$. We leave the details to the reader. \square

We are ready to state the finiteness theorem of local tropicalization.

THEOREM 11.9 (The local finiteness theorem). *Let I be a prime ideal of the ring $K[[\Gamma]]$ of formal power series over an affine pointed semigroup Γ , and assume that the Krull dimension of the quotient ring $K[[\Gamma]]/I$ equals d . Then, the local positive tropicalization $\text{Trop}_{> 0}(I) \subseteq \bar{\sigma}^\circ$ and the local nonnegative tropicalization $\text{Trop}_{\geq 0}(I) \subseteq \bar{\sigma}$ are rational PL conical subspaces of pure dimension d and the closure of $\text{Trop}_{> 0}(I)$ in the space $\bar{\sigma}$ is $\text{Trop}_{\geq 0}(I)$.*

PROOF. If the set $I \cap \Gamma$ is nonempty, it is a prime ideal of the semigroup Γ . Therefore, it must equal $\Gamma \setminus \tau^\perp$ for some face τ of the cone σ . In this case:

$$\text{Trop}_{\geq 0}(I) = \text{Trop}_{\geq 0}(I_\tau),$$

where I_τ is the ideal of $K[[\Gamma_\tau]]$ generated by all τ -truncations f_τ of $f \in I$, and the same for the positive tropicalization. The right hand side of the equality above is a subset of $L(N/N_\tau, \pi_\tau(\sigma))$, which in turn is naturally a subset of $L(N, \sigma)$. The equality itself follows from the commutative diagram:

$$\begin{array}{ccc} K[[\Gamma]]/(\Gamma \setminus \tau^\perp) & \longrightarrow & K[[\Gamma]]/I \\ \parallel & & \parallel \\ K[[\Gamma_\tau]] & \longrightarrow & K[[\Gamma_\tau]]/I_\tau. \end{array}$$

Also notice that our theorem is obvious if $I = \{0\}$.

Thus, in the rest of the proof we assume that I is monomial free and nonzero. Let $\mathcal{B} = \{f_1, \dots, f_m\}$ be an extended tropical basis for I (see Definition 10.4). From Theorem 11.2, Definition 10.1, and Proposition 11.8 we deduce that:

$$(11.1) \quad \text{Trop}_{\geq 0}(I) = \bigcap_{i=1}^m \left(\bigcup_{\tau \leq \sigma} \text{Newton}^\perp((f_i)_\tau) \right).$$

We conclude that $\text{Trop}_{\geq 0}(I)$ is a rational PL conical subspace. By Theorem 11.2 (ii), the same holds for the positive tropicalization. We have:

$$(11.2) \quad \text{Trop}_{> 0}(I) = \left(\bigcap_{i=1}^m \left(\bigcup_{\tau \leq \sigma} \text{Newton}^\perp((f_i)_\tau) \right) \right) \cap \bar{\sigma}^\circ.$$

The last assertion of the theorem also follows directly from (11.1) and (11.2) and the equality $\text{Newton}^\perp(f_i) \cap \bar{\sigma} = \text{Newton}^\perp(f_i)$, where the closure is taken in σ . Thus, it remains only to prove the assertion about the dimension of the local positive tropicalization.

For any $f = \sum a_m \chi^m \in K[[\Gamma]]$ and $w \in N \cap \bar{\sigma}$, consider a deformation:

$$f_t = t^{-w(f)} \sum a_m t^{\langle w, m \rangle} \chi^m \in K[t][[\Gamma]]$$

of f , where $K[t][[\Gamma]]$ is the ring of formal power series over Γ with coefficients in the polynomial ring $K[t]$, and let $I_t \subseteq K[t][[\Gamma]]$ be the ideal generated by all f_t , $f \in I$. Write $S = K[t][[\Gamma]]/I_t$. Notice that for any fixed $t_0 \in K$, $t_0 \neq 0$, the assignment $\chi^m \mapsto t_0^{\langle w, m \rangle} \chi^m$ defines an automorphism of the ring $K[[\Gamma]]$.

Let us show that for every $w \in \text{Trop}_{>0}(I)$, there exist a maximal face of $\text{Trop}_{>0}(I)$ of dimension d containing w . We already know that $\text{Trop}_{>0}(I)$ is a rational PL conical subspace, thus its rational points (with respect to the lattice $N = N(\Gamma)$) are dense in it. Thus, it is sufficient to prove our claim for an integer point $w \in N$. Furthermore, in a neighborhood of the point w the local positive tropicalization of the ideal I coincides with that of the initial ideal $\text{in}_w(I)$ (Remark 11.3).

Consider the deformation $I_t \in K[t][[\Gamma]]$ of the ideal I . By Lemma 11.10 below, we may view $K[t][[\Gamma]]/I_t$ as a flat family of schemes over $\text{Spec } K[t]$. The special fiber over 0 of this family is $\text{Spec } K[[\Gamma]]/\text{in}_w(I)$. Also, the ring $K[t][[\Gamma]]/I_t$ is equidimensional (Lemma 11.12). Now from [Mats, 21.B Theorem 50] it follows that all irreducible components of the special fiber have the same dimension as a general fiber $\text{Spec } K[[\Gamma]]/I$, namely d . In other words, every minimal associated prime of $\text{in}_w(I) \subset K[[\Gamma]]$ has depth d . If $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ are all these minimal primes, then we have the decomposition $\text{Trop}_{>0}(\text{in}_w(I)) = \cup_{1 \leq i \leq k} \text{Trop}_{>0}(\mathfrak{p}_i)$ by Lemma 5.18. Choose \mathfrak{p}^* such that $w \in \text{Trop}_{>0}(\mathfrak{p}^*)$.

Notice that the ideal $J^* = \text{in}_w(I) \subset K[[\Gamma]]$ is generated by power series which are in fact polynomials. Thus we can also consider the ideal J generated by the same polynomials inside the ring $K[\Gamma]$. It is contained in the maximal ideal $\Gamma^+ = \Gamma \setminus \{0\}$, thus by standard theory of completions ([AM, Proposition 10.13, Corollary 11.19] [Nag, Corollary 17.9, 17.12]) we conclude:

$$K[[\Gamma]]/J^* \simeq \widehat{K[\Gamma]/J}, \quad \dim K[\Gamma]/J = \dim K[[\Gamma]]/J^* = d.$$

The positive tropicalization of J^* is just the part of the usual tropicalization of J contained in $\bar{\sigma}^\circ$ (see Proposition 12.5). Therefore, since $\dim K[\Gamma]/J = d$, we already see that the faces of $\text{Trop}_{>0}(J^*)$ passing through w have dimension not greater than d . On the other hand, $\mathfrak{p} = \mathfrak{p}^* \cap K[\Gamma]$ is a prime containing the ideal J , thus $\dim K[\Gamma]/\mathfrak{p} \leq d$. Now, let $\widehat{\mathfrak{p}} = \mathfrak{p}K[[\Gamma]]$. We have again:

$$\dim K[[\Gamma]]/\widehat{\mathfrak{p}} = \dim K[\Gamma]/\mathfrak{p}.$$

But $\widehat{\mathfrak{p}} \subseteq \mathfrak{p}^*$, hence $\dim K[[\Gamma]]/\widehat{\mathfrak{p}} \geq d$. We conclude that $\dim K[\Gamma]/\mathfrak{p} = d$. By properties of the usual tropicalization, $\text{Trop}(\mathfrak{p})$ is purely d -dimensional. Clearly, $w \in \text{Trop}(\mathfrak{p})$. This implies our claim and the theorem. \square

In the following three lemmas we keep the notations of the proof of Theorem 11.9.

LEMMA 11.10 (cf. [Eis, Theorem 15.17]). *The $K[t]$ -algebra S is flat.*

PROOF. Fix a monomial ordering \preceq refining the preorder \preceq_w . By Proposition 9.11, any class of $K[[\Gamma]] \bmod I$ has a unique representative of the form:

$$\sum_{m \in \Gamma} a_m \chi^m,$$

where $a_m = 0$ for each $\chi^m \in \text{in}_{\preceq}(I)$. We claim that $\Gamma \setminus \text{in}_{\preceq}(I)$ plays a similar role for S over $K[t]$. Indeed, if $g \in S$ and $\sum a_m(t) \chi^m$ is a representative of g in $K[t][[\Gamma]]$, let $a_m(t) \chi^m$ be its leading term (with respect to \preceq) with $m \in \text{in}_{\preceq}(I)$. Then, we can find an element $f \in I_t$ with leading term $c \chi^m$, $c \in K^*$. Take the reduction $g - (1/c)a(t)f$. In this way, we delete from g all monomials contained in the initial ideal $\text{in}_{\preceq}(I)$. Moreover, under such a reduction the terms of g that are less than χ^m remain unchanged. Thus, despite the reduction process is infinite, the terms that are less than a given monomial χ^m can change only a finite number of times. This shows that the result of the reduction is an element of $K[t][[\Gamma]]$.

We see, in particular, that S has no torsion as a $K[t]$ -module. Over a principal ideal domain this is equivalent to being flat ([Eis, Corollary 6.3]). \square

LEMMA 11.11. *If $I \subset K[[\Gamma]]$ is a prime ideal, then $I_t \subset K[t][[\Gamma]]$ is also prime.*

PROOF. Assume that $ab \in I_t$, with $a, b \in K[t][[\Gamma]]$:

$$a = \sum a_m(t) \chi^m, \quad b = \sum b_m(t) \chi^m.$$

Fix a monomial ordering refining the w -partial ordering. Substituting $t = 1$ to a and b we get $a(1)b(1) \in I_1 = I$. First assume that $a(1) \neq 0$, $b(1) \neq 0$. Since the ideal I is prime, one of $a(1)$, $b(1)$, say $a(1)$, is contained in I . Consider the deformation $g = (a(1))_t$ of $a(1)$. After a choice of an appropriate coefficient $c(t)$ the first monomial of the the reduction $a - c(t)g$ which is not 0 at $t = 1$ is less than that of a . Notice that $(a - c(t)g)b \in I_t$, thus we may take $a - c(t)g$ instead of a . Repeating this argument, we come to a situation when either one of the series in the product is 0, and thus a or $b \in I_t$, or every coefficient of every term of a or b takes value 0 under the substitution $t = 1$. In this case every coefficient of a (or b) is divisible by $t - 1$. Then we have a relation of the form:

$$(t - 1)^k a' b' \in I_t,$$

where $a'(1)$ and $b'(1) \neq 0$. But since the algebra $S = K[t][[\Gamma]]/I_t$ has a basis consisting of monomials, it follows that $a' b' \in I_t$. Notice that after a finite number of the previous two steps the initial monomial χ^m of a or b will drop with respect to the chosen monomial ordering. After this, new reductions involve only the monomials strictly greater than χ^m . This implies the convergence of the process of reduction of a and b . Thus, $a \in I_t$ or $b \in I_t$, as we wanted to show. \square

LEMMA 11.12. *The ring $K[t][[\Gamma]]/I_t$ is equidimensional, that is, if \mathfrak{m}_1 and \mathfrak{m}_2 are any two maximal ideals in this ring, then the height of \mathfrak{m}_1 equals the height of \mathfrak{m}_2 .*

PROOF. Since the ideal I_t is prime (Lemma 11.11), it suffices to show that the ring $R = K[t][[\Gamma]]$ is equidimensional. Let $\mathfrak{m} \subset R$ be a maximal ideal. The crucial observation is that any series $\sum_{m \in \Gamma} a_m(t)\chi^m$ that begins with a non-zero constant $a_0(t) = a_0 \in K$, $a_0 \neq 0$, is invertible. It follows that:

$$\mathfrak{m}_0 = \{a_0(t) \mid a_0(t) + \sum_{m \in \Gamma^+} a_m(t)\chi^m \in \mathfrak{m}\}$$

is a *proper* ideal of $K[t]$. Moreover, $\mathfrak{m}_0 \neq \{0\}$ because otherwise \mathfrak{m} would not be maximal (a bigger ideal would be, e.g., $\mathfrak{m} + (t)$). Let $\mathfrak{m}' = \mathfrak{m} \cap K[t]$. If $\mathfrak{m}' = \{0\}$, consider the ideal $\mathfrak{m} + (f(t))$, where f generates \mathfrak{m}_0 . It contains \mathfrak{m} as a proper subset. On the other hand, $\mathfrak{m} + (f(t)) = (1)$ is impossible because this would imply that f is invertible. This contradicts the maximality of \mathfrak{m} . Thus, $\mathfrak{m}' = \mathfrak{m}_0 = (f)$, and f is irreducible in $K[t]$. Furthermore, we have $R/Rf \simeq L[[\Gamma]]$, where L is a finite algebraic extension of K . The ideal \mathfrak{m} maps to the maximal ideal of $L[[\Gamma]]$ under the canonical projection $R \rightarrow R/Rf$. Since $L[[\Gamma]]$ is a finite $K[[\Gamma]]$ module, both algebras have the same Krull dimension, equal to the height of their maximal ideals. Let $\dim K[[\Gamma]] = d$. The height of \mathfrak{m} equals the dimension of the localization $R_{\mathfrak{m}}$, and, since $f \in \mathfrak{m}$, $R/Rf \simeq R_{\mathfrak{m}}/R_{\mathfrak{m}}f$. By [AM, Corollary 11.18], we get $\dim R_{\mathfrak{m}}/R_{\mathfrak{m}}f = d + 1$. This number is independent of \mathfrak{m} . \square

COROLLARY 11.13. *Let I be an ideal of the formal power series ring $K[[\Gamma]]$. Then $\text{Trop}_{\geq 0}(I)$ and $\text{Trop}_{> 0}(I)$ are rational PL conical subspaces in $\bar{\sigma}$ and $\bar{\sigma}^\circ$ respectively.*

PROOF. It suffices to consider the nonnegative tropicalization. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be the minimal associated primes of the ideal I . It follows from Lemma 5.18 that:

$$\text{Trop}_{> 0}(I) = \bigcup_{i=1}^k \text{Trop}_{> 0}(\mathfrak{p}_i).$$

But each of $\text{Trop}_{> 0}(\mathfrak{p}_i)$ is a rational PL conical subspace by Theorem 11.9. \square

Now, let us pass to a more general setting, which applies when we aim to tropicalize a family of schemes or varieties over a field K , as explained after the proof of the next theorem:

THEOREM 11.14. *Let Γ be an arbitrary affine pointed semigroup. Let $\gamma: \Gamma \rightarrow (R, \cdot)$ be a local morphism, where (R, \mathfrak{m}) is a complete local ring. Assume that R contains a field K and consider the induced local morphism of rings $\bar{\gamma}: K[[\Gamma]] \rightarrow R$. If R is either:*

- a) *integral over $\bar{\gamma}(K[[\Gamma]])$, or*
- b) *Noetherian, flat over $K[[\Gamma]]$, and the ideal $(\Gamma^+)R$ is prime,*

then:

$$\text{Trop}_{> 0}(\gamma) = \text{Trop}_{> 0}(\ker \bar{\gamma}) \text{ and } \text{Trop}_{\geq 0}(\gamma) = \text{Trop}_{\geq 0}(\ker \bar{\gamma}).$$

In particular, the positive tropicalization $\text{Trop}_{>0}(\gamma)$ is a rational PL conical subspace in $\bar{\sigma}^\circ$, and similarly the nonnegative tropicalization $\text{Trop}_{\geq 0}(\gamma)$ is a rational PL conical subspace in $\bar{\sigma}$.

PROOF. The theorem is a consequence of Theorems 7.2 and 11.9. Indeed, by Theorem 7.2 any local valuation on $K[[\Gamma]]/\ker \bar{\gamma}$ extends to a local valuation on R . On the other hand, any local valuation on R obviously restricts to a local valuation on $K[[\Gamma]]/\ker \bar{\gamma}$. Thus, we have the equality $\text{Trop}_{>0}(\gamma) = \text{Trop}_{>0}(\ker \bar{\gamma})$. The proof for the nonnegative tropicalization is similar. \square

We explain now how tropicalization of families can be studied in the framework of *relative tropicalization*. Let Γ be an affine pointed semigroup and I be an ideal of the ring $K[[\Gamma]]$. Consider also the semigroup $\langle t \rangle = \mathbb{Z}_{\geq 0}$, which will be treated as a multiplicative semigroup generated by t . The corresponding semigroup power series ring with coefficients in the field K is isomorphic to the formal power series ring $K[[t]]$ in one variable t . If $\lambda: \langle t \rangle \rightarrow \Gamma$ is a local semigroup morphism, we get an induced morphism of complete local rings $K[[t]] \rightarrow K[[\Gamma]]/I$ and a linear map $\text{Trop}(\lambda)$ of the positive tropicalizations:

$$\text{Trop}(\lambda): \text{Trop}_{>0}(I) \rightarrow \text{Trop}_{>0}(\langle t \rangle) = \overline{\mathbb{R}}_{>0}.$$

Let $\varphi: L(\sigma, N(\Gamma)) \rightarrow L(\mathbb{R}_{>0}, \mathbb{Z})$ be the linear map inducing $\text{Trop}(\lambda)$. Since $\text{Trop}_{>0}(I)$ is a rational PL conical subspace, for any $a \in \mathbb{Q}_{>0}$ the fiber $(\text{Trop}(\lambda))^{-1}(a)$ is a finite rational polyhedral complex in the linear variety $\varphi^{-1}(a)$. Notice that a valuation on $K[[t]]$ is completely determined by its value on the generator t . Thus, the fiber $(\text{Trop}(\lambda))^{-1}(a)$ admits the following interpretation: it is the tropicalization of the valuations on $K[[\Gamma]]/I$ extending the valuation v on $K[[t]]$ and such that $v(t) = a$.

The fiber $(\text{Trop}(\lambda))^{-1}(+\infty)$ is the local tropicalization of the special fiber of the map $\text{Spec}(K[[\Gamma]]/I) \rightarrow \text{Spec } K[[t]]$ over the unique closed point of $\text{Spec } K[[t]]$. With the notation of Definition 6.10, we can write:

$$(\text{Trop}(\lambda))^{-1}(a) = \text{Trop}_{>0}(\mathcal{V}_{(S, v_a)}(K[[\Gamma]]/I, \mathfrak{m}), \gamma),$$

where S is the image of the ring $K[[t]]$ in $K[[\Gamma]]/I$ under the homomorphism λ , v_a is the valuation of S determined by the condition $v_a(t) = a$, \mathfrak{m} is the maximal ideal of $K[[\Gamma]]/I$, and γ is the natural morphism of semigroups $\gamma: \Gamma \rightarrow K[[\Gamma]]/I$. We conclude that:

$$\text{Trop}_{>0}(\mathcal{V}_{(S, v_a)}(K[[\Gamma]]/I, \mathfrak{m}), \gamma)$$

is a finite rational polyhedral complex, and it has pure dimension $d - 1$ if I is a prime ideal of depth d .

12. Comparison between local and global tropicalization

The aim of this section is to explain that the local tropicalization of the germ at a closed orbit of a subvariety of a toric variety can be obtained as the intersection of the global tropicalization with the linear variety associated to the cone describing the closed orbit.

We start with a subscheme X of an affine toric variety $\text{Spec}(K[\Gamma])$. If the toric variety is not normal, we can always pass to its normalization and lift X to it. By Corollary 7.6 and Lemma 12.1 below, this does not change the tropicalization of X .

LEMMA 12.1. *Let Γ be an affine semigroup and K be a field. Then the integral closure of $K[\Gamma]_{(\Gamma^+)}$ in its field of fractions is $K[\text{Sat}(\Gamma)]_{(\text{Sat}(\Gamma)^+)}$.*

PROOF. It is standard that the integral closure of $K[\Gamma]$ in its field of fractions is $K[\text{Sat}(\Gamma)]$ (see, e.g., [Ful]). By [AM, Proposition 5.12], the integral closure of $K[\Gamma]_{(\Gamma^+)}$ in its field of fractions is the ring of fractions $S^{-1}K[\text{Sat}(\Gamma)]$ of the ring $K[\text{Sat}(\Gamma)]$ with respect to the multiplicative sub-semigroup $S := K[\Gamma] \setminus (\Gamma^+)$. Let us show that *this ring of fractions is equal to the localization $K[\text{Sat}(\Gamma)]_{(\text{Sat}(\Gamma)^+)}$.*

Consider an arbitrary fraction $f/g \in K[\text{Sat}(\Gamma)]_{(\text{Sat}(\Gamma)^+)}$, with the property that $f \in K[\text{Sat}(\Gamma)]$ and $g \in K[\text{Sat}(\Gamma)] \setminus (\text{Sat}(\Gamma)^+)$. We want to prove that there exists $h \in K[\text{Sat}(\Gamma)]$ such that $g \cdot h \in S$. We use the following classical fact: if X_1, \dots, X_l are independent variables and $n \in \mathbb{N}^*$, then:

$$(12.1) \quad \prod_{\underline{c}_i} \left(\sum_{j=1}^l c_{ij} X_j \right) = Q(X_1^n, \dots, X_l^n)$$

where the l -uples $\underline{c}_i = (c_{i1}, \dots, c_{il})$ vary among all possible choices of n -th roots of unity in \mathbb{C}^* , and where $Q \in \mathbb{Z}[X_1, \dots, X_l]$. This can be proven by elementary Galois-type arguments. More precisely, we get a polynomial in the n -th powers of the variables because the left-hand side is invariant under any substitution $X_i \mapsto \eta X_i$, where η is an arbitrary n -th root of unity. The coefficients are integers because we work in an integral extension of \mathbb{Z} , obtained by adjoining the n -th roots of unity, and because the left-hand-side is invariant by all the automorphisms of this extension. Moreover, equation (12.1) shows that Q is a homogeneous polynomial (of degree $D = n^l$) and that it contains one power X_i^D of each variable among its monomials.

Denote by $U(X_1, \dots, X_l) \in \mathbb{C}[X_1, \dots, X_l]$ the product of all linear forms of the left-hand side of (12.1) which are distinct from $X_1 + \dots + X_l$. Since the ring $\mathbb{Z}[X_1, \dots, X_l]$ is factorial, we see that $U(X_1, \dots, X_l) \in \mathbb{Z}[X_1, \dots, X_l]$. Let us rewrite (12.1) in the form:

$$(12.2) \quad (X_1 + \dots + X_l) \cdot U(X_1, \dots, X_l) = Q(X_1^n, \dots, X_l^n).$$

Return now to our polynomial $g \in K[\text{Sat}(\Gamma)] \setminus (\text{Sat}(\Gamma)^+)$. Suppose that there are $l \in \mathbb{N}^*$ non-zero terms in g . Choose an order t_1, \dots, t_l of them, and denote $m_i \in \Gamma$ the exponent of t_i . Replace the variables X_i of (12.2) by the terms t_i . If we choose $n \in \mathbb{N}$ so that $n \cdot m_i \in \Gamma$ for all the exponents m_i of the monomials of g (which is possible by the definition of the saturation), then $Q(t_1^n, \dots, t_l^n) \in K[\Gamma]$. Moreover, we claim that $Q(t_1^n, \dots, t_l^n) \in S = K[\Gamma] \setminus (\Gamma^+)$. If this holds, the proof is finished, as $h = U(t_1^n, \dots, t_l^n)$ satisfies the desired property $g \cdot h \in S$.

Let us explain why $Q(t_1^n, \dots, t_l^n) \in S$. Consider the Newton polyhedron $\mathcal{N}(g) \subset M(\Gamma)_{\mathbb{R}}$ of g , i.e., the convex hull of the exponents of its monomials. The hypothesis that $g \in S$ shows that $\mathcal{N}(g)$ has at least one vertex in Γ^* . Since Γ^* is a face of Γ , there exists $v \in N(\Gamma)$ which, when seen as

a function on the vertices of $\mathcal{N}(g)$, attains its minimum on exactly one vertex, which is moreover contained in Γ^* . Assume that it is the vertex m_1 . Then, the exponent Dm_1 appears in $Q(t_1^n, \dots, t_l^n)$ only once, coming from the monomial X_1^D of $Q(X_1^n, \dots, X_l^n)$. Indeed, suppose that $X_1^{a_1} \cdots X_l^{a_l}$ is any other monomial of $Q(X_1^n, \dots, X_l^n)$. The exponent of the term $t_1^{a_1} \cdots t_l^{a_l}$ of $K[\Gamma]$ is $a_1m_1 + \cdots + a_lm_l$. As Q is homogeneous of degree D , we have $\sum_{i=1}^l a_i = D$. Therefore:

$$\langle v, a_1m_1 + \cdots + a_lm_l - Dm_1 \rangle = \sum_{i=2}^l a_i \langle v, m_i - m_1 \rangle.$$

Our hypothesis that the new monomial is distinct from X_1^D shows that at least one of the nonnegative integers a_2, \dots, a_l is positive. Choose such an $a_k > 0$. Since also $\langle v, m_k - m_1 \rangle > 0$ and all the other members a_i and $\langle v, m_i - m_1 \rangle$ in this formula are nonnegative, we conclude that the exponent of $t_1^{a_1} \cdots t_l^{a_l}$ is indeed different from the exponent of t_1^D . Therefore, $Q(t_1^n, \dots, t_l^n) \in S$, as it contains the monomial t_1^D . \square

Thus, there is no loss in generality if we assume in this section that Γ is a *saturated* affine semigroup. Denote $\Gamma = \text{Sat}(\Gamma) = \check{\sigma} \cap M(\Gamma)$. If R is a ring and \mathfrak{p} a prime ideal, let $\psi_{\mathfrak{p}}$ denote the associated morphism of localization $\psi_{\mathfrak{p}}: R \rightarrow R_{\mathfrak{p}}$. The proofs of the following results are easy and left to the reader.

LEMMA 12.2. *Let $\Gamma \xrightarrow{\gamma} (R, \cdot)$ be a morphism of semigroups and let $I(\gamma) \subset R$ be the ideal generated by the image $\gamma(\Gamma^+)$. Let \mathfrak{p} be a prime ideal of R containing $I(\gamma)$ (that is, a point of the subscheme of $\text{Spec } R$ defined by the ideal $I(\gamma)$). Then the morphism of semigroups $\Gamma \xrightarrow{\psi_{\mathfrak{p}} \circ \gamma} (R_{\mathfrak{p}}, \cdot)$ satisfies $\psi_{\mathfrak{p}} \circ \gamma(\Gamma^+) \subset \mathfrak{p}R_{\mathfrak{p}}$.*

PROPOSITION 12.3. *Let R be a ring, \mathfrak{p} be one of its prime ideals and $(\Gamma, +) \xrightarrow{\gamma} (R, \cdot)$ a morphism of semigroups such that $\gamma(\Gamma^+) \subset \mathfrak{p}$. Then, for any subspace $\mathcal{W} \subset \mathcal{V}(R)$, we have:*

$$\text{Trop}(\mathcal{W}, \gamma) \cap \bar{\sigma}^\circ(\Gamma) = \text{Trop}_{>0}(\mathcal{V}(\psi_{\mathfrak{p}})^{-1}(\mathcal{W}), \gamma).$$

In particular, we get the following property of subschemes of toric varieties, comparing local and global tropicalization:

COROLLARY 12.4. *Let X be a subscheme of a toric variety $\mathcal{Z}(\Delta, N)$. Let $A \in X$ be a closed point which is an orbit O_σ of $\mathcal{Z}(\Delta, N)$, where σ is a cone of Δ with non-empty interior. Then: $\text{Trop}_{>0}(X, A) = \text{Trop}(X) \cap \bar{\sigma}^\circ$.*

We would like to emphasize the special case used in the proof of Theorem 11.9 (which holds for arbitrary, not necessarily saturated, pointed affine semigroups):

PROPOSITION 12.5. *Let Γ be an affine pointed semigroup, let I be an ideal of the ring $K[\Gamma]$ contained in the maximal ideal (Γ^+) , and let \hat{I} be the extension of I in the power series ring $K[[\Gamma]]$. Then: $\text{Trop}_{>0}(\hat{I}) = \text{Trop}(I) \cap \bar{\sigma}^\circ$.*

PROOF. If a ring valuation v of $K[\Gamma]$ is nonnegative on Γ and positive on Γ^+ , then it is nonnegative on the whole ring $K[\Gamma]$ and positive on the maximal ideal (Γ^+) . Thus, the valuation v canonically extends to a local valuation of the ring $K[[\Gamma]]$. Conversely, any local valuation w of $K[[\Gamma]]$ restricts to a nonnegative valuation of $K[\Gamma]$, which is positive on the maximal ideal (Γ^+) . \square

In fact, we can reconstruct the global tropicalization of a subvariety or a subscheme X over a field K of a toric variety $\mathcal{Z}(\Delta, N)$ from the local tropicalizations of the germs of this subscheme at the orbits of some birational modification of $\mathcal{Z}(\Delta, N)$. If X does not pass through any such orbit (e.g., $X = 1 \in \mathbb{T}$), then the global tropicalization of X consists of one point and there is essentially nothing to reconstruct. So, let us suppose that this is not the case.

NOTATION 12.6. Let σ be a cone of Δ , and O_σ the corresponding orbit of the big torus in $\mathcal{Z}(\Delta, N)$. O_σ is the unique closed orbit of the affine toric variety $\mathcal{Z}(\sigma, N) = \text{Spec } K[\check{\sigma} \cap M]$. We denote the semigroup $\check{\sigma} \cap M$ by Γ . Assume that the orbit O_σ is contained in the subscheme X . Let $I_{X,\sigma}$ denote the ideal of X in the local ring $K[\Gamma]_{(\Gamma^+)}$, and $\widehat{I_{X,\sigma}}$ the corresponding ideal in the completion $K(\Gamma^*)[[\Gamma']]$ of $K[\Gamma]_{(\Gamma^+)}$ at its maximal ideal (see Section 8). We have the positive tropicalization $\text{Trop}_{>0}(\widehat{I_{X,\sigma}}) = \text{Trop}_{>0}(X, \sigma)$, which is a PL conical subspace in $\bar{\sigma}^\circ$, and the nonnegative tropicalization $\text{Trop}_{\geq 0}(\widehat{I_{X,\sigma}}) = \text{Trop}_{\geq 0}(X, \sigma)$, which is a PL conical subspace in $\bar{\sigma}$. These tropicalizations are well defined due to the following result:

PROPOSITION 12.7. *Let I be an ideal of the power series ring $K[[\Gamma]]$, where K is an arbitrary field. Let Φ be an automorphism of $K[[\Gamma]]$ sending each element of Γ to a product of itself by a unit of $K[[\Gamma]]$. Then, the positive and the nonnegative tropicalizations of I and of $\Phi(I)$ coincide.*

PROOF. If v is any nonnegative ring valuation of $K[[\Gamma]]$, then $v(u) = 0$ for any unit u of $K[[\Gamma]]$. It follows that $v(\Phi(f)) = v(f)$ for all $f \in K[[\Gamma]]$. \square

Proposition 12.7 shows that the local tropicalization of a germ of subvariety of an affine toric variety at the unique closed orbit depends only on the *toroidal structure* in the neighborhood of that orbit. In Section 13 we will use this fact to define tropicalization of subvarieties of algebraic toroidal embeddings. As a first application of the previous proposition, we generalize Proposition 12.5.

PROPOSITION 12.8. *Let Γ be a saturated affine semigroup, and I an ideal of $K[\Gamma]$ contained in the ideal (Γ^+) . Fix an isomorphism $\widehat{K[\Gamma]_{(\Gamma^+)}} \simeq K(\Gamma^*)[[\Gamma']]$ and let \widehat{I} be the extension of I in the ring $K(\Gamma^*)[[\Gamma']]$. Then:*

$$\text{Trop}_{>0}(\widehat{I}) = \text{Trop}(I) \cap \bar{\sigma}^\circ.$$

PROOF. Recall that an isomorphism between the completion of $K[\Gamma]_{(\Gamma^+)}$ and $K(\Gamma^*)[[\Gamma']]$ is defined up to a unit. By Proposition 12.7, the positive tropicalization $\text{Trop}_{>0}(\widehat{I})$ does not depend on the isomorphism between $\widehat{K[\Gamma]_{(\Gamma^+)}}$ and $K(\Gamma^*)[[\Gamma']]$. Then the proof goes along the same lines as the proof of Proposition 12.5. \square

Now let X be a subscheme of a toric variety $\mathcal{Z}(\Delta, N)$. We use Notation 12.6.

LEMMA 12.9. *Let τ be a face of σ . Assume that $O_\tau \subseteq X$. Then:*

$$\text{Trop}_{\geq 0}(X, \sigma) \cap (\bar{\tau}^\circ) = \text{Trop}_{> 0}(X, \tau),$$

or, equivalently:

$$\overline{\text{Trop}_{> 0}(X, \sigma)} \cap (\bar{\tau}^\circ) = \text{Trop}_{> 0}(X, \tau).$$

PROOF. Let $\Gamma = \check{\sigma} \cap M$, $\Gamma(\tau) = \check{\tau} \cap M(\Gamma)$. We have the following diagram of rings and ideals:

$$\begin{array}{ccccccc} K(\Gamma^*)[[\Gamma']] & \xleftarrow{a} & K[\Gamma]_{(\Gamma^+)} & \xrightarrow{c} & K[\Gamma]_{(\Gamma \setminus \Gamma_\tau)} & \xrightarrow{b} & K(\Gamma(\tau)^*)[[\Gamma(\tau)']] \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \widehat{I_{X, \sigma}} & \longleftarrow & I_{X, \sigma} & \longrightarrow & I_{X, \tau} & \longrightarrow & \widehat{I_{X, \tau}} \end{array}$$

where c is the morphism of localization, a is the composition of the natural morphism of a local ring to its completion with the fixed isomorphism $\widehat{K[\Gamma]_{(\Gamma^+)}} \simeq K(\Gamma^*)[[\Gamma']]$. b is defined similarly to a , and the arrows in the second row are induced by the arrows in the first.

Now, let \bar{v} be a valuation of the ring $K(\Gamma^*)[[\Gamma']]$ (infinite on the ideal $\widehat{I_{X, \sigma}}$) inducing an element $w \in \text{Trop}_{\geq 0}(X, \sigma) \cap (\bar{\tau}^\circ)$. Let v be the restriction of \bar{v} to $K[\Gamma]_{(\Gamma^+)}$. Since v takes only value 0 on the subsemigroup χ^{Γ_τ} , we can push it forward to the localization $K[\Gamma]_{(\Gamma \setminus \Gamma_\tau)}$ and, since v is positive on the ideal $\chi^{(\Gamma \setminus \Gamma_\tau)}$, we can further push it forward to a local valuation of $K(\Gamma(\tau)^*)[[\Gamma(\tau)']]$, thus producing an element of $\text{Trop}_{> 0}(X, \tau)$. Going to the opposite direction, we can easily show that any local valuation of $K(\Gamma(\tau)^*)[[\Gamma(\tau)']]$ (infinite on the ideal $\widehat{I_{X, \tau}}$) defines a nonnegative valuation on $K(\Gamma^*)[[\Gamma']]$, positive on the ideal $(\Gamma \setminus \Gamma_\tau)$ and trivial on the subsemigroup χ^{Γ_τ} . \square

As a consequence of the results of this section, we get the following theorem describing the connection between the global tropicalization of a subvariety or a subscheme X of a normal toric variety $\mathcal{Z}(\Delta, N)$ and the local tropicalizations of germs of X at the orbits of $\mathcal{Z}(\Delta, N)$.

THEOREM 12.10. *Let Δ be a fan. Let X be a subscheme of the toric variety $\mathcal{Z}(\Delta, N)$ and $\text{Trop}(X) \subseteq L(\Delta, N)$ be the tropicalization of $X \subseteq \mathcal{Z}(\Delta, N)$ in the sense of Remark 6.5. If σ is a cone of Δ such that $O_\sigma \subseteq X$, then:*

$$\text{Trop}(X) \cap \check{\sigma} = \text{Trop}_{> 0}(X, \sigma).$$

By Corollary 7.6 and Lemma 12.1, we get the following generalization of the previous theorem to subschemes of arbitrary, not necessarily normal, toric varieties:

THEOREM 12.11. *Let \mathcal{S} be a fan of semigroups, with associated fan Δ . Let X be a subscheme of the toric variety $\mathcal{Z}(\mathcal{S})$ and $\text{Trop}(X) \subseteq L(\Delta, N)$ be*

the tropicalization of $X \subseteq \mathcal{Z}(\mathcal{S})$ in the sense of Remark 6.5. If σ is a cone of Δ such that $O_\sigma \subseteq X$, then:

$$\text{Trop}(X) \cap \check{\sigma} = \text{Trop}_{>0}(X, \sigma).$$

If the orbit O_σ is not contained in X , then it is natural to set by definition $\text{Trop}_{\geq 0}(X, \sigma) = \text{Trop}_{>0}(X, \sigma) = \emptyset$. Let us consider a particular case when $\mathcal{Z}(\Delta, N) = \mathbb{T}$ is simply a torus. For any subvariety $X \subseteq \mathbb{T}$ we have the familiar tropicalization $\text{Trop}(X)$. In addition $\Gamma = M(\Gamma)$, $\Gamma^+ = (0)$, $\mathbb{T} = \text{Hom}(\Gamma, K^*)$, and $\Delta = (0)$. Then, $K(\Gamma^*)[[\Gamma]] = K(\Gamma)$ is the field of rational functions on \mathbb{T} . If $X = \mathbb{T}$, then $I_X = \{0\}$ and the positive and the nonnegative tropicalization consist of the point $\{0\}$ corresponding to the trivial valuation on $K(\Gamma)$. If X is a proper subvariety, then $\text{Trop}_{\geq 0}(X, 0) = \text{Trop}_{>0}(X, 0) = \emptyset$. Still, the tropicalization $\text{Trop}(X)$ can be reconstructed from local tropicalizations with a help of an auxiliary fan.

Some new terminology and notation is in order. Let Σ be a PL cone (Definition 11.4) in an \mathbb{R} -vector space V and v a point of Σ . If $\Sigma = \cup \sigma$ is a fan structure on Σ , let $\sigma(v)$ be the unique cone that contains v in its relative interior. For the point $v \in \Sigma$ there is a unique subspace $T_v \Sigma \subseteq V$ with the following property: $T_v \Sigma$ is the minimal (with respect to inclusion) subspace of V such that for any fan structure $\Sigma = \cup \sigma$, $T_v \Sigma$ contains the cone $\sigma(v)$. We say that $T_v \Sigma$ is the *tangent space* to Σ at the point v . Now, let Δ be a fan in V . Again, for a point $v \in \text{Supp } \Delta$, we let $\delta(v)$ be the unique cone of Δ such that v is contained in the relative interior of $\delta(v)$. We say that a PL cone Σ and a fan Δ are *transversal* at a point $v \in \text{Supp } \Delta \cap \Sigma$ if $T_v \Sigma + \langle \delta(v) \rangle = V$, where $\langle \delta(v) \rangle$ is the subspace of V spanned by $\delta(v)$. We say that Σ and Δ are transversal if they are transversal at each point $v \in \text{Supp } \Delta \cap \Sigma$.

COROLLARY 12.12. *Let X be a subvariety of a torus $\mathbb{T} = \text{Hom}(\Gamma, K^*)$. Let Δ be a rational polyhedral fan in $N(\Gamma)_{\mathbb{R}}$ that is transversal to the tropicalization $\text{Trop}(X)$ of X and such that $\text{Trop}(X)$ is contained in $\text{Supp } \Delta$. Then, $\text{Trop}(X)$ is a disjoint union of the real parts of all local positive tropicalizations $\text{Trop}_{>0}(X, \sigma)$, $\sigma \in \Delta$.*

PROOF. We shall only outline the main ideas in the proof, leaving the details to the reader. It suffices to show that for each point v of $\text{Trop}(X)$, the closure of X in the toric variety $\mathcal{Z}(\Delta, N)$ contains the orbit $O_{\sigma(v)}$. The ideal $I_{X, \sigma(v)}$ of the closure of X in the affine toric variety $\mathcal{Z}(\sigma(v), N)$ is generated by all polynomials $f \in I_X$ whose support is contained in $\check{\sigma}(v)$. A sufficient condition for such a polynomial $f = \sum a_m \chi^m$ to vanish on $O_{\sigma(v)}$ is that the extended Newton diagram $\text{Newton}^+(f)$ is not generated by one point, i.e., there is no $m \in \Gamma$ such that $\text{Newton}^+(f) = m + \check{\sigma}(v)$. But this condition indeed holds for each $f \in I_{X, \sigma(v)}$ because $v \in \text{Trop}(X)$ and $\text{Trop}(X)$ and Δ are transversal. \square

13. Toroidal meaning of local tropicalization

In this section, we show that tropicalization is an invariant of the ambient toroidal structure. More precisely, the tropicalization of an algebraic, analytic or formal germ of subvariety of an affine toric variety at its closed

orbit, depends only on the associated toroidal structure. We use this fact to define the tropicalization of a subvariety of a toroidal embedding.

The basic reference for the notions used in this section is [KKMS, Chapter II]. First, we recall the basic definitions and fix the notations. The ground field K will be assumed to be algebraically closed.

DEFINITION 13.1. ([KKMS, Chapter II, Definition 1]). A **toroidal embedding** over a field K is a pair (U, X) , where $U \subseteq X$ is a Zariski open subset of a normal algebraic variety X over K , such that for every closed point $x \in X$ there exists an affine toric variety $(\mathbb{T}, \mathcal{Z})$ over K , where \mathbb{T} is the open torus $\mathbb{T} \subseteq \mathcal{Z}$, a closed point $t \in \mathcal{Z}$, and an isomorphism of K -local algebras:

$$\widehat{\mathcal{O}}_{X,x} \simeq \widehat{\mathcal{O}}_{\mathcal{Z},t}$$

such that the ideal in $\widehat{\mathcal{O}}_{X,x}$ generated by the ideal of $X \setminus U$ maps isomorphically to the ideal in $\widehat{\mathcal{O}}_{\mathcal{Z},t}$ generated by the ideal of $\mathcal{Z} \setminus \mathbb{T}$.

Notice that the previous definition implies that U is smooth. The notation (U, X) , with U coming first, is intended to suggest that X is thought of as a total space into which U embeds and that, as for toric varieties, this total space may change without changing U .

The orbit of $t \in \mathcal{Z}$ can always be assumed closed, by diminishing perhaps \mathcal{Z} . Such a pair (\mathcal{Z}, t) , together with a formal isomorphism as above is called a *local model* of (U, X) at x . The definition implies that the irreducible components of $X \setminus U$ (if nonempty) have codimension 1 in X . We denote them by $(E_i)_{i \in I}$, so that $X \setminus U = \cup_{i \in I} E_i$. If all the varieties E_i are normal, a toroidal embedding (U, X) is called a *toroidal embedding without self intersections*.

In the sequel we consider only toroidal embeddings without self intersections. The set U and the connected components of the sets $\cap_{i \in J} E_i \setminus \cup_{i \notin J} E_i$, $J \subseteq I$, define a natural *stratification* of the space X . If Y is a stratum, the *star* $\text{Star}(Y)$ of Y is the union of all strata Z such that Y is contained in the closure of Z .

Let Y be a stratum. Following [KKMS, Chapter II, Definition 3], we denote:

- M^Y = the group of Cartier divisors on $\text{Star}(Y)$, supported on the hypersurface $\text{Star}(Y) \setminus U$;
- $N^Y = \text{Hom}(M^Y, \mathbb{Z})$;
- M_+^Y = subsemigroup of M^Y of effective divisors;
- $\sigma^Y = \{w \in N_{\mathbb{R}}^Y \mid \langle w, u \rangle \geq 0 \text{ for all } u \in M_+^Y\} \subseteq N_{\mathbb{R}}^Y$.

Note that the cone σ^Y is strongly convex and that $\text{rk } M^Y = \text{codim}_X Y$.

PROPOSITION 13.2. *Let Y be a stratum of the toroidal embedding without self-intersection (U, X) . Then, the completion $\widehat{\mathcal{O}}_{X,Y}$ of the local ring of X at Y is isomorphic to the ring $K(Y)[[M_+^Y]]$ of formal power series over the semigroup M_+^Y with coefficients in the field $K(Y)$ of rational functions on Y . This isomorphism is defined up to multiplication by units.*

PROOF. Each Cartier divisor on $\text{Star}(Y)$ defines a principal ideal in the local ring $\mathcal{O}_{X,Y}$. Thus, to each element of M_+^Y we may assign a defining

function, i.e., an element of $\mathcal{O}_{X,Y}$ (well-defined up to a unit). We Define this correspondence on a set of elements of M_+^Y which form a basis of $M_{\mathbb{R}}^Y$ and then extend it to all of M_+^Y . We obtain a morphism of semigroups $M_+^Y \rightarrow \mathcal{O}_{X,Y}$. Notice also that the ring $\mathcal{O}_{X,Y}$ and hence its completion $\widehat{\mathcal{O}}_{X,Y}$, contain a field, say, the field K . Then, it follows from the theory of complete rings (see [Nag, Chapter V]) that $\widehat{\mathcal{O}}_{X,Y}$ contains also a field isomorphic to its residue field, that is to $K(Y)$. Let us fix such a subfield. In this way, we get a morphism of rings:

$$\alpha_Y: K(Y)[[M_+^Y]] \rightarrow \widehat{\mathcal{O}}_{X,Y},$$

unique up to a unit. We now prove that it is an isomorphism of complete local rings.

The injectivity of α_Y is clear, so let us prove the surjectivity. Since $\mathcal{O}_{X,Y}$ naturally embeds into its completion $\widehat{\mathcal{O}}_{X,Y}$ and the ring $K(Y)[[M_+^Y]]$ is complete, it suffices to prove that $\mathcal{O}_{X,Y}$ lies in the image of $K(Y)[[M_+^Y]]$. First, note that the image of M_+^Y generates the maximal ideal of $\mathcal{O}_{X,Y}$. Indeed, consider the diagram:

$$\mathcal{O}_{X,Y} \hookrightarrow \mathcal{O}_{X,x} \hookrightarrow \widehat{\mathcal{O}}_{X,x} \xrightarrow{\varphi} \widehat{\mathcal{O}}_{\mathcal{Z},t}$$

of rings, where x is a closed point of the stratum Y and (\mathcal{Z}, t) is a local model at x . By the properties of toroidal embeddings (see [KKMS, Chapter II, Corollary 1]) the ideal of the stratum Y maps to the ideal of the closed orbit of \mathcal{Z} under φ . This last ideal is generated by the image of M_+^Y in $\widehat{\mathcal{O}}_{\mathcal{Z},t}$. Let \mathfrak{m} be the maximal ideal of $\mathcal{O}_{X,Y}$. We see that M_+^Y is a subset of \mathfrak{m} and it generates the ideal $\widehat{\mathfrak{m}}$ of the stratum Y in $\widehat{\mathcal{O}}_{X,x}$. But since the ring $\mathcal{O}_{X,x}$ is Noetherian, we conclude that M_+^Y also generates \mathfrak{m} .

Consider now some $f \in \mathcal{O}_{X,Y}$. Fix a finite subset $\{f_1, \dots, f_k\}$ of M_+^Y which generates the maximal ideal \mathfrak{m} . Let $a_0 \in K(Y)$ be a representative of the class of f in $\mathcal{O}_{X,Y}/\mathfrak{m}$. Then, $f - a_0 \in \mathfrak{m}$ and we can write:

$$f - a_0 = \sum_i g_i f_i, \quad g_i \in \mathcal{O}_{X,Y} \text{ for all } i.$$

Applying the same argument to g_i we find $a_1, \dots, a_k \in K(Y)$ such that:

$$f = a_0 + \sum_i a_i f_i \pmod{\mathfrak{m}^2}.$$

Repeating this argument we represent f as an image of a series in $K(Y)[[M_+^Y]]$. This proves that α_Y is surjective, as we wanted to show. \square

To each toroidal embedding, we canonically associate a *conical polyhedral complex with integral structure*. Let us recall the construction.

DEFINITION 13.3. ([KKMS, Chapter II, Definition 5]). A **conical polyhedral complex** Δ is formed by:

- a topological space $|\Delta|$;
- a finite family of closed subsets σ_i called **cones**;
- a finite dimensional real vector space V_i of real valued continuous functions on σ_i such that:

- (1) a basis of V_i defines a homeomorphism from σ_i to a polyhedral cone $\sigma'_i \subset \mathbb{R}^{n_i}$, not contained in a hyperplane;
- (2) faces of σ'_i correspond also to cones of Δ ;
- (3) $|\Delta|$ is a disjoint union of relative interiors of σ_i for all i ;
- (4) if σ_j is a face of σ_i , then the restriction of V_i to σ_j is V_j .

REMARK 13.4. Even if we use the same notation as for fans of cones, it is important to note that in a conical polyhedral complex we do not have an embedding of the various cones in a fixed vector space. In particular, if we consider the conical polyhedral complex associated to a fan, we loose the information about this embedding.

DEFINITION 13.5. ([KKMS, Chapter II, Definition 6]). An **integral structure** on a conical polyhedral complex Δ is a set of finitely generated abelian groups $L_i \subset V_i$ such that:

- (1) $(L_i)_{\mathbb{R}} \simeq V_i$;
- (2) if σ_j is a face of σ_i , then the restriction of L_i to σ_j is L_j .

Let (U, X) be a toroidal embedding. Let Y be a stratum, and Z a stratum in $\text{Star}(Y)$. Then, the canonical surjective map $M^Y \rightarrow M^Z$ induces a canonical inclusion $N_{\mathbb{R}}^Z \rightarrow N_{\mathbb{R}}^Y$ such that $N^Z = N_{\mathbb{R}}^Z \cap N^Y$, and if Z corresponds to the face τ of σ^Y , then the inclusion $N_{\mathbb{R}}^Z \rightarrow N_{\mathbb{R}}^Y$ maps σ^Z isomorphically to τ (see [KKMS, Chapter II, Corollaries 1 and 2] for the details). Now consider the topological space:

$$|\Delta| = \bigsqcup_Y \sigma^Y / \sim,$$

where the disjoint union is taken over all strata of (U, X) and the equivalence relation \sim is the gluing of cones along common faces. The triple $(|\Delta|, M_{\mathbb{R}}^Y, M^Y)$ is called the conical polyhedral complex (simply *conical complex* in the sequel) of the toroidal embedding (U, X) .

For each cone σ^Y of the conical complex Δ we have a linear variety $L(\sigma^Y, N^Y)$ and the closure $\overline{\sigma^Y}$ (see Section 4). *The gluing of cones of Δ naturally extends to a gluing of their closures.* More precisely, let Y_1 , Y_2 , and Z be strata of (U, X) , and suppose that Y_1 and Y_2 are contained in the closure of Z . Recall that $\overline{\sigma^{Y_1}}$ is defined as the set of all nonnegative simigroup homomorphisms from $\sigma^{Y_1} \cap M^{Y_1}$ to $\overline{\mathbb{R}}$, and similarly $\overline{\sigma^{Y_2}}$ and $\overline{\sigma^Z}$. Since M^Z is naturally a sublattice of both M^{Y_1} and M^{Y_2} , and σ^Z is a common face of σ^{Y_1} and σ^{Y_2} , $\text{Hom}_{sg}(\sigma^Z \cap M^Z, \overline{\mathbb{R}})$ is a common subset of $\text{Hom}_{sg}(\sigma^{Y_1} \cap M^{Y_1}, \overline{\mathbb{R}})$ and $\text{Hom}_{sg}(\sigma^{Y_2} \cap M^{Y_2}, \overline{\mathbb{R}})$. This allows to glue the extended cones $\overline{\sigma^{Y_1}}$ and $\overline{\sigma^{Y_2}}$ along $\overline{\sigma^Z}$. The stratum at infinity of $\overline{\sigma^{Y_1}}$ that corresponds to the face σ^Z is equipped with the lattice N^{Y_1}/N^Z and the vector space $(N^{Y_1}/N^Z)_{\mathbb{R}}$. For an illustration in dimension two, see Example 13.10 and the accompanying Figure 6.

DEFINITION 13.6. Let Δ be the conical complex of a toroidal embedding (U, X) . Denote by $|\overline{\Delta}| = (\bigsqcup_Y \overline{\sigma^Y}) / \sim$ the topological space obtained by gluing the extended cones of Δ as explained before. Equip it with the additional structure M^Y that is inherited from Δ , and with all the analogous additional structure (quotient lattices, vector spaces of real functions) on the

strata of $\overline{\sigma^Y}$ at infinity. We call it the **extended conical complex** of the toroidal embedding (U, X) , denoted $\overline{\Delta}$.

Now, let \mathcal{I} be an ideal sheaf on a toroidal embedding (U, X) defining a subscheme W . This sheaf generates an ideal I^Y (perhaps non-proper) in the local ring $\mathcal{O}_{X,Y}$ of every stratum Y . Fix an isomorphism $\widehat{\mathcal{O}}_{X,Y} \simeq K(Y)[[M_+^Y]]$ and let \widehat{I}^Y be the ideal generated by I^Y in $K(Y)[[M_+^Y]]$. Let Γ be the semigroup M_+^Y and γ the natural morphism of semigroups:

$$\gamma: M_+^Y \rightarrow K(Y)[[M_+^Y]]/\widehat{I}^Y.$$

Then, we have the positive tropicalization $\text{Trop}_{>0}(W, Y) = \text{Trop}_{>0}(\gamma)$ and the nonnegative tropicalization $\text{Trop}_{\geq 0}(W, Y) = \text{Trop}_{\geq 0}(\gamma)$, which are conical sets in $(\overline{\sigma^Y})^\circ$ respectively in $\overline{\sigma^Y}$. By Proposition 12.7, these tropicalizations do not depend on the choice of an isomorphism between $\widehat{\mathcal{O}}_{X,Y}$ and $K(Y)[[M_+^Y]]$.

LEMMA 13.7. *Let Y and Z be strata, and $Z \subseteq \text{Star}(Y)$. If $Z \subseteq W$, then:*

$$\text{Trop}_{\geq 0}(W, Y) \cap (\overline{\sigma^Z})^\circ = \text{Trop}_{>0}(W, Z).$$

PROOF. The proof is essentially the same as the proof of Lemma 12.9. \square

This lemma justifies the following definition:

DEFINITION 13.8. Let W be a subscheme of a toroidal embedding (U, X) . The disjoint union:

$$\text{Trop}(W) = \bigsqcup_Y \text{Trop}_{>0}(W, Y)$$

of positive tropicalizations of all germs of W at strata of (U, X) , considered as a subset of the extended conical complex $\overline{\Delta}$ of the toroidal embedding (U, X) is called the **tropicalization** of the subscheme W .

THEOREM 13.9. *Let W be a subscheme of a toroidal embedding (U, X) . Then for every stratum Y of (U, X) the intersection $\text{Trop}(W) \cap \overline{\sigma^Y}$ is a rational polyhedral conical set. If the germ of W at Y has pure dimension d , then $\text{Trop}(W) \cap \overline{\sigma^Y}$ has pure real dimension d .*

PROOF. The proof follows from Theorem 11.9. \square

EXAMPLE 13.10. In the top part of Figure 6 is represented a (singular) curve W in a smooth surface X , and E_1, \dots, E_4 are smooth curves of X crossing normally in succession at the points A, B, C . Therefore, if $U := X \setminus \bigcup_{1 \leq i \leq 4} E_i$, the pair (X, U) is a toroidal embedding. In the bottom part of the figure we represent the associated tropicalisation, which is obtained by gluing the positive local tropicalisations in the neighborhood of the points A, B, C . We denote by σ_P the 2-dimensional cone corresponding to each point $P \in \{A, B, C\}$, and by τ_i the 1-dimensional cone corresponding to the curve E_i , for each $i \in \{1, \dots, 4\}$. Notice that at the point C we have two irreducible components of W , but that their tropicalizations coincide, as both are smooth and transversal to E_3 and E_4 .

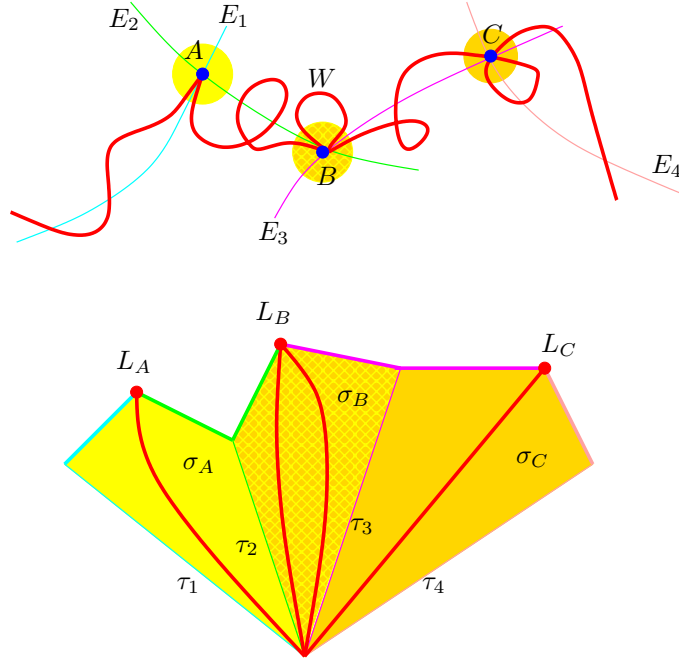


FIGURE 6. Tropicalization of a curve in a toroidal embedding of dimension 2

14. An extension of the definition of tropicalization

There is yet a more general version of local tropicalization. We are not going to develop the theory here, but, as we promised in Introduction, we shall describe the main idea of the construction.

If R is any commutative ring with unit and (R^*, \cdot) its group of units, then, as explained after Definition 2.8, we can define the quotient R/R^* as a multiplicative semigroup. Any nonnegative valuation v on R defines a semigroup morphism $R/R^* \rightarrow \mathbb{R}_{\geq 0}$ (the argument is the same as the one given in the proof of Proposition 12.7). Then, we can speak about tropicalization of subsets $\mathcal{W} \subset \mathcal{V}(R)$ not only in the presence of semigroup morphisms $\gamma: \Gamma \rightarrow R$, but also of morphisms defined modulo units, that is of semigroup morphisms:

$$\gamma: \Gamma \rightarrow R/R^*.$$

This yields a functorial construction that generalizes the one described in Section 6.

For instance, let η be a point (not necessarily closed) of a normal algebraic variety (over an arbitrary algebraically closed field K) or an analytic space X . From now on, we consider X as a germ at the point η . Let $D = \cup D_i$ be a reduced hypersurface on X . *We do not assume that the pair (X, D) is toroidal in any sense.* Let Γ be the semigroup of effective Cartier divisors supported on D . The semigroup Γ generalizes of the semigroup M_+^Y defined in Section 13. Since the semigroup of *all* effective Cartier divisors on X is isomorphic to $\mathcal{O}_{X,\eta}/\mathcal{O}_{X,\eta}^*$, the semigroup Γ is naturally embedded

in $\mathcal{O}_{X,\eta}/\mathcal{O}_{X,\eta}^*$. This embedding is given by assigning to each Cartier divisor a defining function (well-defined modulo a unit).

Let us show that Γ is an affine semigroup. Denote by G the group of all Weil divisors supported on D , and by H the group of all Cartier divisors supported on D . The group G is free, thus H is free as a subgroup of G . All effective Weil \mathbb{Q} -divisors form a rational polyhedral cone σ of maximal dimension in $G_{\mathbb{Q}}$. Thus $\Gamma = H \cap \sigma$ is finitely generated by Gordan's lemma ([Ful, Section 1.2, Proposition 1]). We conclude that Γ is indeed an affine semigroup. Therefore, whenever a hypersurface D on a normal germ X is fixed, we can tropicalize any ideal I of the local ring $\mathcal{O}_{X,\eta}$, by considering either the positive or the nonnegative local tropicalization of the canonical map $\Gamma \rightarrow R/R^*$, where $R := \mathcal{O}_{X,\eta}/I$.

In this way, we extend the notions of positive and nonnegative local tropicalizations to the case of local semigroup morphisms $(\Gamma, +) \rightarrow (R/R^*, \cdot)$, where R is an arbitrary local ring.

15. Comparison with the literature

In this section we compare our work with other results in the literature, we sketch some possible directions of development and we conclude by stating two open problems.

There are already several books and plenty of papers on tropical geometry. The field is developing very fast, and sometimes ideas come to minds of several authors almost simultaneously. It may well happen that our work is very close to something already done or something currently being developed by other researchers. In this section we would like to explain what we think is new in our approach and what is taken from other sources.

The idea of tropicalization, though the term itself is relatively new, appeared already in Bergman's paper [Berg2] from 1971. Even all three definitions of the tropicalization (using valuations, the definition based on initial ideals, and the one using K -valued points) are present there. Bieri and Groves [BG] proposed the elegant point of view that the piecewise-linear complexes that are now called tropicalizations are *invariants of the morphisms $M \rightarrow K^*$ from a finitely generated free abelian group M to the multiplicative group K^* of a field K* or, more generally, of the morphisms $M \rightarrow (R, \cdot)$ to the multiplicative semigroup of a ring R .

As the reader should remember, we defined local tropicalization as a subset of an extended affine space, and this subset corresponds to a morphism $\Gamma \rightarrow (R, \cdot)$ from a semigroup Γ . This generalizes Bieri and Groves' point of view, though Payne's work [Pay08], where tropicalizations of embeddings into arbitrary toric varieties are studied, was also very motivating for us. Extensions of affine spaces (called linear varieties in our paper) were already defined in [AMRT]. They are explained also in [Kaj], [Pay08] and [Rab]; our presentation has no substantial differences, but we describe in more detail the topology of those spaces.

As far as we know, tropicalizations of semigroup morphisms $\Gamma \rightarrow R$ for arbitrary local rings R have not been studied in the literature before.

However, tropicalizations of not only algebraic but also analytic objects were defined and studied by Touda [Tou], Rabinoff [Rab], and Gubler [Gub].

In fact, the main part of our paper (Sections 8, 11, and 12) were an extension of Touda's work [Tou], though we started this project without knowing about it. Touda studies tropicalizations of ideals in the ring of formal power series over the field \mathbb{C} of complex numbers. He works with the definition of local tropicalization using weights (analog of the second one used for global tropicalization, as recalled in the introduction). He proves then a theorem about piecewise-linear structure of the local tropicalization. As an important tool in his proofs, he uses the notion of local Gröbner fan of an ideal in a formal power series ring, as well as its properties proven by Bahloul and Takayama in [BT 04].

The differences with our approach are the following. We work in the more general setting of morphisms $\Gamma \rightarrow (R, \cdot)$, in particular, R can be an algebra over an arbitrary field K , and we consider general ring valuations which lead to local tropicalizations living in an extended affine space, whereas Touda restricts only to the real part of the local tropicalization. Another new result in our local finiteness theorem is the statement about dimension of local tropicalization. We should also note that some important steps of the construction of a tropical basis (e.g., [Tou, Proposition 6.3]) are left without proof in [Tou].

The main objects of the papers [Rab] and [Gub] are rings of series with some convergence conditions over fields endowed with a *nontrivial* valuation and ideals in these rings. Notice that our local conditions (see Definition 6.6) imply that if the local ring R has a subfield K , then any local valuation on R is *trivial* on K . Thus we think that our work is in a way complementary to [Gub] and [Rab]. Another important difference is that we could work completely without the theory of affinoid algebras that plays a major role in [Gub] and [Rab], and in the proof of piecewise-linear structure of the tropicalization in [EKL]. The local conditions lead also naturally to the question about extensions of nonnegative valuations treated in Section 7. Despite the fact that the literature on the valuation theory is very rich, we are not aware of any reference for questions of this kind.

In the proof of the local finiteness theorem we follow well-known ideas. The use of Gröbner basis techniques in describing the structure of tropicalization is common, perhaps, since the paper [SS] of Speyer and Sturmfels. To show the existence of universal standard, or Gröbner, bases in power series rings we apply the method of Sikora [Sik] (as explained by Boldini in [Bol09]). Different and more constructive proofs should exist, but we do not know about them. It would be interesting to check if Sikora's method is applicable also to affinoid algebras. As it is said in [Rab, Remark 8.8], a theorem on the existence of a universal standard basis for an ideal in an affinoid algebra would be an important part of the analytic tropical geometry. The method of a flat degeneration of an ideal to its initial ideal is rather standard, see, e.g., [Eis, Theorem 15.17]. The fact that an ideal I and its initial ideal $\text{in}_w(I)$ locally around w have the same tropicalization has also been observed earlier, see [Rab, Remark 7.9.2].

As we showed in Section 12, the usual tropicalization of subvarieties of a torus or of a toric variety can be glued from the local tropicalizations. However, to claim that our local tropicalization generalizes the usual one would not be completely honest, since we essentially use properties of the tropicalization of subvarieties of toric varieties in the proof of Theorem 11.9.

We are not aware of any other treatment of tropicalization of subvarieties of toroidal embeddings. A new feature in this case is the absence of the “big torus” in a toroidal embedding. However, our local tropicalization is well suited for this situation since it uses only the “formal torus embedding” $\mathrm{Spec} K[[x_1, \dots, x_n]]$. Once the theory of tropicalization of ideals of the rings $K[[\Gamma]]$ has been developed, the construction of tropicalization of subvarieties of toroidal embeddings is very natural and straightforward.

Let us describe now some possible interactions of our work with developing parts of mathematics.

One should be able to prove in the toroidal setting an analog of Payne’s main theorem from [Pay08] relating tropicalizations and analytifications in the Berkovich sense. This would allow to make a bridge with Thuillier’s work [Thu] on the analytification of toroidal embeddings.

Our final general definition of tropicalization associated to a morphism of semigroups $\Gamma \rightarrow R/R^*$ should be useful as a starting point for tropicalizing log-structures. This seems to be one of the current directions of development of tropical geometry, as indicated by Gross in his book [Gr] and in his talk [Gr-talk]. Indeed, a log-scheme is a scheme X equipped with a morphism of sheaves of (multiplicative) semigroups $\alpha_X: \mathcal{M}_X \rightarrow \mathcal{O}_X$, such that α_X realizes an isomorphism between $\alpha_X^{-1}(\mathcal{O}_X^*)$ and \mathcal{O}_X^* . Let $\overline{\mathcal{M}}_X := \mathcal{M}_X / \alpha_X^{-1}(\mathcal{O}_X^*)$. Quoting from [Gr, Page 101]: “*The sheaf of monoids $\overline{\mathcal{M}}_X$, written additively [...] should be viewed as containing combinatorial information about the log structure*”. Note that α_X induces a canonical morphism of sheaves of semigroups:

$$\overline{\mathcal{M}}_X \rightarrow \mathcal{O}_X / \mathcal{O}_X^*.$$

That is, we are ready for gluing our affine definitions of tropicalizations!

The fact that we have isolated the category of semigroups as part of the structure allowing tropicalization should allow us to also make connections with algebraic geometry over the field with one element, as described for instance by Connes and Consani in [CC]. As explained in Chapter 3 of that paper, the category of semigroups and morphisms of semigroups is an essential component of it.

Another field which has already very important connections with tropical geometry is the theory of Berkovich analytic spaces. As explained by Berkovich [Berk-talk], the category of semigroups also plays an important role there. As the title of Berkovich’s talk indicates, this should be seen as part of a project of relating analytic geometry to geometry over the field with one element.

We finish with two problems about local tropicalization.

PROBLEM 15.1. Let $\gamma: (\Gamma, +) \rightarrow (R/R^*, \cdot)$ be an arbitrary local morphism, where Γ is a pointed affine semigroup and R is a complete local ring. We do not suppose that γ is the natural morphism of Γ to a quotient of

a power series ring $K[[\Gamma]]$ over a field K , as in Section 11. Does the local tropicalization $\text{Trop}_{>0}(\gamma)$ have piecewise-linear structure in such a general case? This question is interesting both in the case when R contains a field or when it does not.

PROBLEM 15.2. Find a proof of Theorem 11.9 that is independent of the standard theory of tropicalization of subvarieties of toric varieties.

References

- [Ar] Aroca, F., *Krull tropical hypersurfaces*. Ann. Fac. Sci. Toulouse Math. (6) **19** (2010), 525–538.
- [AMRT] Ash, A., Mumford, D., Rapoport, M., Tai, Y.-S., *Smooth compactifications of locally symmetric varieties*. Math. Sci. Press, 1975. Second revised edition published by Cambridge Univ. Press, 2010.
- [AM] Atiyah, M. F., Macdonald, I. G., *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969.
- [BT 04] Bahloul, R., Takayama, N., *Local Gröbner fan: polyhedral and computational approach*, ArXiv:math/0412044.
- [BT 07] Bahloul, R., Takayama, N., *Local Gröbner fans*, C. R. Acad. Sci. Paris, Ser. I **344** (2007), 147–152.
- [Berg1] Bergman, G. M., *A weak Nullstellensatz for valuations*, Proc. Amer. Math. Soc. **28** (1971), 32–38.
- [Berg2] Bergman, G. M., *The logarithmic limit-set of an algebraic variety*, Trans. Amer. Math. Soc. **157** (1971), 459–469.
- [Berk] Berkovich, V. G., *Spectral theory and analytic geometry over non-Archimedean fields*. Mathematical Surveys and Monographs, **33**. American Mathematical Society, Providence, RI, 1990.
- [Berk-talk] Berkovich, V. G., *Analytic geometry over \mathbb{F}_1* . Talk given on the 4th of February 2011 at the conference “From p-adic differential equations to arithmetic algebraic geometry”, Padova. http://www.wisdom.weizmann.ac.il/~vova/Padova-slides_2011.pdf
- [BG] Bieri, R., Groves, J.R.J., *The geometry of the set of characters induced by valuations*, J. Reine Angew. Math. **347** (1984), 168–195.
- [BJSST] Bogart, T., Jensen, A. N., Speyer, D., Sturmfels, B., Thomas, R. R., *Computing tropical varieties*. J. Symb. Comp. **42** (2007), no. 1-2, 54–73.
- [Bol09] Boldini, R., *Universal Gröbner bases in polynomial rings*, preprint, 2009. <http://www.math.uzh.ch/index.php?file&key1=13499>
- [Bol10] Boldini, R., *A topological approach to leading monomial ideals*. ArXiv:1008.0286v2.
- [CC] Connes, A., Consani, C., *Characteristic one, entropy and the absolute point*, in Non-commutative Geometry, Arithmetics, and Related Topics, Proc. of the JAMI Conference 2009, JHUP (2012), 75–140.
- [CLOS 97] Cox, D., Little, J., O’Shea, D., *Ideals, varieties and algorithms*. Second Edition, Springer, 1997.
- [CLOS 05] Cox, D., Little, J., O’Shea, D., *Using Algebraic Geometry*. Second Edition, Springer, 2005.
- [CLS] Cox, D., Little, J., Schenck, H. K., *Toric varieties*. Springer, 2011.
- [D] Draisma, J., *A tropical approach to secant dimensions*, J. Pure and Appl. Algebra **212** (2008), 349–363.
- [EKL] Einsiedler, M., Kapranov, M., Lind, D., *Non-archimedean amoebas and tropical varieties*, J. Reine Angew. Math. **601** (2006), 139–157.
- [Eis] Eisenbud, D., *Commutative Algebra with a View Toward Algebraic Geometry*. Springer, 2004.
- [FJT] Fukuda, K., Jensen, A. N., Thomas, R. R., *Computing Gröbner fans*, Math. Comp. **76** No. 260 (2007), 2189–2212.
- [Ful] Fulton, W., *Introduction to Toric Varieties*, Princeton University Press, 1993.

- [GT] González Pérez, P., Teissier, B., *Toric geometry and the Semple-Nash modification*. RACSAM, 2012. DOI 10.1007/s13398-012-0096-0.
- [Gr] Gross, M., *Tropical geometry and mirror symmetry*. A. M. S., 2011.
- [Gr-talk] Gross, M., *The logarithmic view of tropical geometry*. Talk at the Tropical Geometry Workshop, CIEM, Castro Urdiales, 16th of December 2011.
- [Gub] Gubler, W., *Tropical varieties for non-Archimedean analytic spaces*, Invent. Math. **169** (2007), 321–376.
- [Gub 11] Gubler, W., *A guide to tropicalizations*, in *Tropical Geometry*, Proceedings Castro Urdiales 2011, E. Brugallé, M.A. Cueto, A. Dickenstein, E.M. Feichtner and I. Itenberg editors, Contemp. Mathematics, A. M. S., 2013, this volume.
- [Hir] Hironaka, H., *Resolution of singularities of an algebraic variety over a field of characteristic zero: II*. Annals of Math. **79** No. 2 (1964), 205–326.
- [HY] Hocking, J. G., Young, G. S., *Topology*. Constable and Company, Ltd., 1961, reprinted by Dover Publications, 1988.
- [How] Howie, J. M., *Fundamentals of semigroup theory*. Clarendon Press, Oxford, 1995.
- [IMS] Itenberg, I., Mikhalkin, G., Shustin, E., *Tropical algebraic geometry*. Oberwolfach Seminars **35**, Birkhäuser Verlag, 2007.
- [JMM] Jensen, A. N., Markwig, H., Markwig, T., *An algorithm for lifting points in a tropical variety*. Collect. Math. **59** (2008), no. 2, 129–165.
- [JP] De Jong, T., Pfister, G., *Local analytic geometry*. Vieweg, 2000.
- [Kaj] Kajiwara, T., *Tropical toric geometry*, in *Toric topology*, Contemp. Math., **460** (2008), 197–207.
- [KKMS] Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B., *Toroidal embeddings I*, LNM 339, 1973, Springer-Verlag.
- [Mats] Matsumura, H., *Commutative algebra*, W. A. Benjamin Co., New York, 1970.
- [MR] Mora, T., Robbiano, L., *The Gröbner fan of an ideal*. J. Symb. Comput. **6** (2-3) (1988), 183–208.
- [Nag] Nagata, M., *Local rings*, Interscience Tracts in Pure and Appl. Math. **13**, J. Wiley, New York, 1962.
- [Pay07] Payne, S., *Fibers of tropicalizations*, Math. Z. **262** (2009), no. 2, 302–311.
- [Pay08] Payne, S., *Analytification is the limit of all tropicalizations*, Math. Res. Lett. **16** (2009) no.3, 543–556.
- [Rab] Rabinoff, J., *Tropical analytic geometry, Newton polygons, and tropical intersections*, Adv. Math. **229** (2012), 3192–3255.
- [Shi] Shibuta, T., *Irreducibility criterion for algebroid curves*, Math. Comp. **82** (2013), 531–554.
- [Sik] Sikora, A. S., *Topology on the spaces of orderings of groups*. Bull. London Math. Soc. **36** (2004), 519–526.
- [Sp] Speyer, D., *Tropical Geometry*, PhD thesis, University of California, Berkeley, 2005.
- [SS] Speyer, D., Sturmfels, B., *The tropical Grassmannian*. Adv. Geom. **4** (2004), no. 3, 389–411.
- [ST] Sturmfels, B., Tevelev, J., *Elimination theory for tropical varieties*. Math. Res. Lett. **15** (2008), no. 3, 543–562.
- [Tei] Teissier, B., *Amibes non-archimédiennes*. In *Géométrie tropicale*, Les Éditions de l'École Polytechnique, 2008, 85–114.
- [Tev] Tevelev, J., *Compactifications of subvarieties of tori*, Amer. J. Math. **129** (4) (2007), 1087–1104.
- [Thu] Thuillier, A., *Géométrie toroïdale et géométrie analytique non archimédienne. Application au type d'homotopie de certains schémas formels*. Manuscripta Math. **123** (2007), no. 4, 381–451.
- [Tou] Touda, N., *Local tropical variety*, ArXiv:math/0511486.
- [V] Vauquié, M., *Valuations*. (French) Resolution of singularities (Obergrugl, 1997), 539–590, Progr. Math., 181, Birkhäuser, Basel, 2000.
- [ZS] Zariski, O., Samuel, P., *Commutative Algebra*. Vol. II, D. Van Nostrand Company, Inc., 1960.

UNIVERSITÉ LILLE 1, UFR DE MATHS., BÂTIMENT M2, CITÉ SCIENTIFIQUE, 59655,
VILLENEUVE D'ASCQ CEDEX, FRANCE.

E-mail address: `patrick.popescu@math.univ-lille1.fr`

BAUMAN MOSCOW STATE TECHNICAL UNIVERSITY, DEP. OF MATHEMATICAL MOD-
ELING, UL. 2-YA BAUMANSKAYA 5, MOSCOW 105005, RUSSIA

E-mail address: `dstepanov@bmstu.ru`